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Free vibration analysis of two-dimensional structures using Coons-patch macroelements

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Abstract

Recently, Coons' interpolation was used for the construction of large finite elements with degrees of freedom appearing mostly along the boundaries of a structure. In the regime of elasticity problems, these so-called "Coons-patch macroelements" were successfully applied to the static analysis of plane structures [C.G. Provatidis, Analysis of axisymmetric structures using Coons' interpolation, Finite Elem. Anal. Des. 39 (2003) 535–558.] while this paper continues the research by investigating their performance in the extraction of natural frequencies and mode shapes. Apart from the piecewise-linear and cubic B-splines interpolation previously used, the performance of Lagrange polynomials and the role of additional internal nodes is studied here. Relationships with classical Serendipity and Lagrangian type elements are discussed. Moreover, the capability of Coons-patch macroelements to couple with conventional finite elements is investigated. The proposed method was applied to three illustrative examples and it was successfully compared with conventional bilinear finite elements.

Keywords: Large elements; Macroelements; Coons interpolation; CAD/CAE; Natural frequencies; Two-dimensional structures

1. Introduction

A lot of attempts have been made within the last years in order to replace conventional finite element methods with other methods such as the boundary element method (BEM) [1] or mesh-free and meshless techniques [2–5]. Essentially, the main practical need that justifies the relevant research activity is to minimize data preparation cost (related to the time-consuming mesh generation task) and to increase the accuracy in calculations by a simultaneous reduction of analysis effort. However, so far BEM did not achieve to replace FEM in the regime of dynamic analysis since its original formulation suffers from frequency-dependent fundamental solutions. This fact leads to a nonalgebraic problem while its alternative dual reciprocity formulation (DR/BEM) [6] highly depends on the choice of the radial basis functions or requires internal nodes [7–9]. Moreover, the mesh-free techniques are not always shortcomings-free due to difficulties related to the inversion of the matrix of coefficients [10]. Therefore, the need of a robust and effective computational technique is still timely.

During the last six years, the above thoughts have motivated the author to develop a new method for the construction of large finite elements with the nodal points along the boundaries only. The background of the method is Coons' interpolation, a formula established in CAD-surface theory that was applied to the automotive industry of USA since the middle 1960s. In the framework of engineering analysis, this method has been successfully applied mainly to potential [11–14] and recently to static elasticity problems [15].

Since not adequate experience exists regarding the behavior of Coons-patch macroelements (CPM) in elastodynamics, this paper aims to further investigate their capability of solving structural free vibration problems (eigenfrequency and mode shape extraction) and compare with conventional FEM solutions of the same boundary discretization. So far, the CPM approach has been applied in conjunction with piecewise-linear and cubic B-spline interpolation along the boundary of an elastic structure. In addition to that, this paper extends the latter interpolation to also Lagrange polynomials of which numerical integration aspects and size limitations are discussed.

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Moreover, the necessity of using internal nodes is investigated and a systematic procedure of dealing with them is proposed.

2. Formulation of Coons-patch macroelements (CPM)

2.1. Macroelements using only boundary nodes

As the basic theory has been previously presented [15], in the sequence only the essential parts are outlined. Two-dimensional CPM treat the entire problem domain, or a large portion of that, as a four-sided patch ABCD on the (x, y)-plane. The real patch is mapped to a reference patch (ξ, η), where the normalized curvilinear coordinates vary between 0 and 1 $(0 \leqslant \xi, \eta \leqslant 1)$ as shown in Fig. 1. According to Coons' interpolation formula, each point $\mathbf{x}(\xi, \eta) = \{x(\xi, \eta),\ y(\xi, \eta)\}^{\mathrm{T}}$ in the patch can be approximated by its boundaries $(\mathbf{x}(\xi, 0),\ \mathbf{x}(\xi, 1),\ \mathbf{x}(0, \eta),\ \mathbf{x}(1, \eta))$ as follows:

$$\begin{aligned}\mathbf{x}(\xi, \eta) = {} & E_0(\xi)\mathbf{x}(0, \eta) + E_1(\xi)\mathbf{x}(1, \eta) + E_0(\eta)\mathbf{x}(\xi, 0) \\ & + E_1(\eta)\mathbf{x}(\xi, 1) - E_0(\xi)E_0(\eta)\mathbf{x}(0, 0) \\ & - E_1(\xi)E_0(\eta)\mathbf{x}(1, 0) - E_0(\xi)E_1(\eta)\mathbf{x}(0, 1) \\ & - E_1(\xi)E_1(\eta)\mathbf{x}(1, 1),\end{aligned} \tag{1}$$

where the blending functions can be chosen, for example, to be linear as follows:

$$\begin{aligned}& E_0(\xi) = 1 - \xi, \quad E_1(\xi) = \xi, \\ & E_0(\eta) = 1 - \eta, \quad E_1(\eta) = \eta.\end{aligned} \tag{2}$$

Now, the idea of *isoparametric* elements is applied to Eq. (1) for the interpolation of the displacement vector $\mathbf{u}(\xi, \eta) = \{u(\xi, \eta), v(\xi, \eta)\}^{\mathrm{T}}$ within the patch, as follows:

$$\begin{aligned}\mathbf{u}(\xi, \eta) = {} & E_0(\xi)\mathbf{u}(0, \eta) + E_1(\xi)\mathbf{u}(1, \eta) + E_0(\eta)\mathbf{u}(\xi, 0) \\ & + E_1(\eta)\mathbf{u}(\xi, 1) - E_0(\xi)E_0(\eta)\mathbf{u}(0, 0) \\ & - E_1(\xi)E_0(\eta)\mathbf{u}(1, 0) - E_0(\xi)E_1(\eta)\mathbf{u}(0, 1) \\ & - E_1(\xi)E_1(\eta)\mathbf{u}(1, 1).\end{aligned} \tag{3}$$

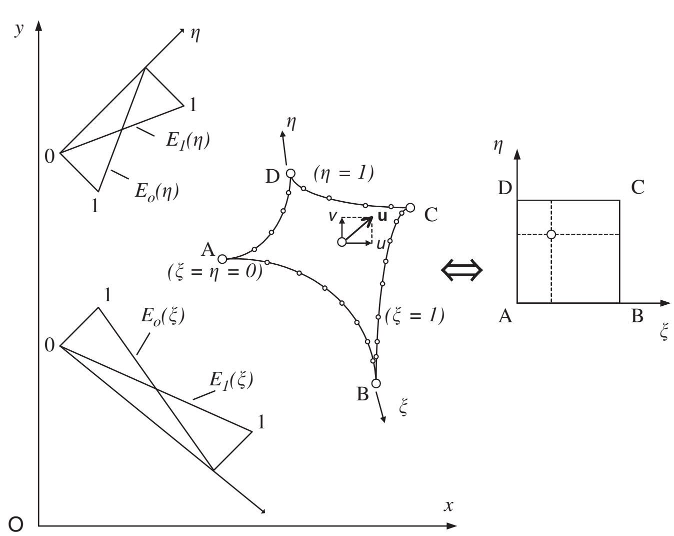

Fig. 1. Definition of a Coons-patch macroelement.

Let us assume that the sides AB, BC, CD and DA include q_1, q_2, q_3 and q_4 nodes, respectively. Then, the total number of nodes along the boundary of the patch becomes

$$q_e = q_1 + q_2 + q_3 + q_4 - 4. \tag{4}$$

If the boundary values $\mathbf{u}(\xi, 0)$, $\mathbf{u}(\xi, 1)$, $\mathbf{u}(0, \eta)$ and $\mathbf{u}(1, \eta)$ in (3) are interpolated by any set of trial functions $B_j(\hat{\xi})$ ($\hat{\xi}$ is either ξ or η; the upper index in B_j below corresponds to the relevant side):

$$\begin{aligned}&\text{side AB}: \ \mathbf{u}(\xi, 0) = \sum_{j=1}^{q_1} B_j^{\mathrm{AB}}(\xi)\ \mathbf{u}(\xi_j, 0), \\ &\text{side BC}: \ \mathbf{u}(1, \eta) = \sum_{j=1}^{q_2} B_j^{\mathrm{BC}}(\eta)\ \mathbf{u}(1, \eta_j), \\ &\text{side CD}: \ \mathbf{u}(\xi, 1) = \sum_{j=1}^{q_3} B_j^{\mathrm{CD}}(\xi)\ \mathbf{u}(\xi_j, 1), \\ &\text{side DA}: \ \mathbf{u}(0, \eta) = \sum_{j=1}^{q_4} B_j^{\mathrm{DA}}(\eta)\ \mathbf{u}(0, \eta_j),\end{aligned} \tag{5}$$

and then Eq. (3) is collocated to all boundary nodes of the reference macro-element, the global cardinal shape functions $N_j(\xi, \eta)$ within the Coons-patch can be finally constructed, so that the solution $\mathbf{u}(\xi, \eta)$ is approximated by

$$\mathbf{u}(\xi, \eta) = \sum_{j=1}^{q_e} N_j(\xi, \eta)\ \mathbf{u}_j(t), \tag{6}$$

with $\mathbf{u}_j(t)$ denoting time-dependent displacement at nodal point 'j', appearing at the boundaries of the macro-element. The previous procedure leads to the following expressions for the global shape functions:

(i) Corner nodes:

$$\begin{aligned}N_{\mathrm{A}}(\xi, \eta) &= E_0(\xi)B_{q_4}^{\mathrm{DA}}(\eta) + E_0(\eta)B_1^{\mathrm{AB}}(\xi) - E_0(\xi)E_0(\eta), \\ N_{\mathrm{B}}(\xi, \eta) &= E_1(\xi)B_1^{\mathrm{BC}}(\eta) + E_0(\eta)B_{q_1}^{\mathrm{AB}}(\xi) - E_1(\xi)E_0(\eta), \\ N_{\mathrm{C}}(\xi, \eta) &= E_1(\xi)B_{q_2}^{\mathrm{BC}}(\eta) + E_1(\eta)B_1^{\mathrm{CD}}(\xi) - E_1(\xi)E_1(\eta), \\ N_{\mathrm{D}}(\xi, \eta) &= E_0(\xi)B_1^{\mathrm{DA}}(\eta) + E_1(\eta)B_{q_3}^{\mathrm{CD}}(\xi) - E_0(\xi)E_1(\eta).\end{aligned} \tag{7}$$

(ii) Interior nodes to AB (local numbering):

$$N_j(\xi, \eta) = E_0(\eta)B_j^{\mathrm{AB}}(\eta), \quad 2 \leqslant j \leqslant q_1 - 1. \tag{8}$$

(iii) Interior nodes to BC (local numbering):

$$N_j(\xi, \eta) = E_1(\xi)B_j^{\mathrm{BC}}(\eta), \quad 2 \leqslant j \leqslant q_2 - 1. \tag{9}$$

(iv) Interior nodes to CD (local numbering):

$$N_j(\xi, \eta) = E_1(\eta)B_j^{\mathrm{CD}}(\xi), \quad 2 \leqslant j \leqslant q_3 - 1. \tag{10}$$

(v) Interior nodes to DA (local numbering):

$$N_j(\xi, \eta) = E_0(\xi)B_j^{\mathrm{DA}}(\eta), \quad 2 \leqslant j \leqslant q_4 - 1. \tag{11}$$