Polymer 106 (2016) 14–20

Contents lists available at ScienceDirect

Polymer

journal homepage: www.elsevier.com/locate/polymer

On the constant parameters of Halpin-Tsai equation

Mahmood M. Shokrieh*, Hadi Moshrefzadeh-Sani

Composites Research Laboratory, Center of Excellence in Experimental Solid Mechanics and Dynamics, School of Mechanical Engineering, Iran University of Science and Technology, Tehran, 16846-13114, Iran

ARTICLE INFO

Article history:
Received 3 August 2016
Received in revised form
24 October 2016
Accepted 25 October 2016
Available online 26 October 2016

Keywords:
Halpin-Tsai method
Laminated analogy approach
Randomly oriented composites

ABSTRACT

The Halpin-Tsai method is a well-known technique to calculate the stiffness of composites reinforced by micro and nano particles. In this method, first the longitudinal and transverse moduli of composites are calculated. Then, the elastic modulus of randomly oriented composites is obtained using an equation contains a constant coefficient. This coefficient is assumed to be constant and independent of the matrix and reinforcement properties. The aim of the present research is to calculate this coefficient with an analytical model and show that it depends on the matrix and reinforcement properties. In this regard, an analytical method called the Mori-Tanaka laminated analogy (MT-LA) was developed which is able to calculate the elastic modulus of the randomly-oriented composites. Comparing the result of the MT-LA method with that of the Halpin-Tsai equation, the coefficient of Halpin-Tsai equation was obtained. It was shown that this coefficient is not constant and depends on the volume fraction and the stiffness ratio of the matrix to reinforcement. Finally, using this new coefficient, equations are presented which are able to compute the elastic modulus of both platelet and fibrous randomly oriented composites. Using the mechanical properties of the carbon nanotube (CNT), graphene sheet (GS) and the polymer, the elastic moduli of nanocomposites were calculated. The results were compared with experimental data available in the literature. It was shown that more accurate results were achieved by using this new coefficient in the Halpin-Tsai equation.

1. Introduction

The Halpin-Tsai [1] and modified role of mixture [2] methods are known as two most straightforward ways to calculate the elastic modulus of the randomly oriented composites. These methods are widely used to calculate the modulus of nanocomposites reinforced by various nano particles [3–5]. The Halpin-Tsai method is very popular in both micro and nano mechanics because of its simplicity [6–8]. In fact, this method is suggested to calculate the longitudinal and transvers properties of the aligned reinforcement composites and the stiffness of the randomly oriented composites is usually calculated as $aE_L + (1-a)E_T$, where E_L and E_T are the longitudinal and transvers moduli of the iso-oriented composites, respectively and a is a constant coefficient. The coefficient a is considered to be equal to 3/8 when the fibers are dispersed two-dimensionally [2]. van Es et al. [9] found that the value of the coefficient a is 0.184 when the fibers are dispersed three-dimensionally. For the composites with the platelet-reinforcement the coefficient a is considered to be equal to 0.49 [10], however some researchers used a equal to 3/8 for this kind of composites [11–13]. The CNTs are dispersed three dimensionally in the matrix but some researchers [14] also used $a = 3/8$ in their modeling, although this value were suggested for the composites with 2-D dispersion of the reinforcement.

To the best knowledge of the present authors, the coefficient a is always assumed to be constant and independent of the matrix and reinforcement properties. It seems that there is a lack of analytical studies to calculate this coefficient. In this paper to obtain the value of this coefficient, a new analytical method (MT-LA) is developed which is able to calculate the elastic modulus of the randomly oriented composites directly. The MT-LA is a combination of the Mori-Tanaka [15] and the laminated-analogy [16] methods. Comparing the results of the MT-LA method with the Halpin-Tsai equation, the new value of the coefficient a is obtained. It was shown that the coefficient a is not constant and depends on the volume fraction and the ratio of the stiffness of the matrix to that of the reinforcement. The calculated results according to this non-constant coefficient and the Halpin-Tsai method were compared

---

* Corresponding author. Tel.: +98 21 77208127.
E-mail address: Shokrieh@iust.ac.ir (M.M. Shokrieh).

http://dx.doi.org/10.1016/j.polymer.2016.10.049
0032-3861/© 2016 Elsevier Ltd. All rights reserved.

with some experimental data available in the literature. It was shown that more accurate results were achieved by the Halpin-Tsai model with non-constant coefficient.

2. The Halpin-Tsai method

The Halpin-Tsai method is a semi-empirical method, which usually uses to calculate the elastic modulus of both fibrous and platelet reinforcement composites. This method is very popular in both micro and nano mechanics because of its simplicity. In this method, the longitudinal and transverse moduli (E_L and E_T) of composites were calculated. For the fibrous-composites, E_L and E_T are obtained from the following equations [1]:

$$E_L = E_m \frac{1 + \frac{l}{r}\eta_L V_f}{1 - \eta_L V_f}, E_T = E_m \frac{1 + 2\eta_T V_f}{1 - \eta_T V_f}$$
$$\eta_L = \frac{\frac{E_r}{E_m} - 1}{\frac{E_r}{E_m} + \frac{l}{r}}, \eta_T = \frac{\frac{E_r}{E_m} - 1}{\frac{E_r}{E_m} + 2} \tag{1}$$

For the platelet-reinforcement composites E_L and E_T are obtained from the following equations [1]:

$$E_L = E_m \frac{1 + 2\frac{l}{t}\eta_L V_f}{1 - \eta_L V_f}, E_T = E_m \frac{1 + 2\eta_T V_f}{1 - \eta_T V_f}$$
$$\eta_L = \frac{\frac{E_r}{E_m} - 1}{\frac{E_r}{E_m} + 2\frac{l}{t}}, \eta_T = \frac{\frac{E_r}{E_m} - 1}{\frac{E_r}{E_m} + 2} \tag{2}$$

where E_m and E_r are the elastic moduli of the matrix and reinforcement, respectively Also, l is the length of the reinforcement. Moreover, r and t are the radius and thickness of the fibrous and platelet reinforcement, respectively. The elastic modulus of randomly-oriented composites (E_C) was obtained using the following equation [2]:

$$E_C = aE_L + (1 - a)E_T \tag{3}$$

where traditionally $a = 0.375$ (i.e., 3/8) for 2-D and $a = 0.184$ for 3-D dispersion of the fibrous reinforcements [9]. To the best knowledge of the present authors, the coefficient a is always assumed to be constant. In this paper, using the MT-LA, the non-constant value of the coefficient a is explored for different volume fractions and stiffness ratios of the reinforcement and matrix.

3. The Mori-Tanaka laminated analogy (MT-LA) approach

In this paper the Mori-Tanaka [15] and the laminated analogy [16] methods were coupled together to calculate the elastic modulus of the randomly oriented composites. In this regard, the Mori-Tanaka method is used to calculate the stiffness matrix of the aligned reinforcement composites and then the stiffness of the randomly oriented composites is calculated by the laminated analogy approach.

Some specific assumptions are assumed here:

- The effect of nano particle waviness was neglected.
- Both nanoparticle and matrix are linear elastic.
- Perfect bond exists between the nano particles and matrix.
- There is no residual stress between the nano particles and matrix.

One of the main limitations of the present method is neglecting the agglomeration of the nano particles. So, the modeling was done for nanocomposites with volume fraction less than 1%.

- The Mori-Tanaka method

The Mori-Tanaka method is one of the advanced micromechanical method which is able to calculate the stiffness tensor of composites with different reinforcement shapes. The effect of collective interactions of reinforcements is considerable in this method and it is known as a common way to model the nanocomposites [17–19]. In this method, it is assumed that each inclusion is embedded in an infinite pristine matrix subjected to an effective strain ε_m or an effective stress σ_m in the far field, where ε_m and σ_m denote the average strain and the average stress over the matrix, respectively. The fourth-order tensor $\boldsymbol{A}$ relates the average strain of the reinforcement ε_r and matrix ε_m via $\varepsilon_r = \boldsymbol{A}{:}\varepsilon_m$ and it is given by:

$$\boldsymbol{A} = \left(\boldsymbol{I} + \boldsymbol{S} : (\boldsymbol{C_m})^{-1} : (\boldsymbol{C}_r - \boldsymbol{C_m})\right)^{-1} \tag{4}$$

where $\boldsymbol{S}$ is the Eshelby tensor. The Eshelby tensor was calculated analytically for the ellipsoid shape reinforcements, which is well documented in Ref. [20]. The effective stiffness tensor $\boldsymbol{C}$ of the aligned reinforcement composites of the same shape is given by:

$$\boldsymbol{C} = \left(V_f \boldsymbol{C_m} + V_f \boldsymbol{C}_r : \boldsymbol{A}\right) : \left(\left(1 - V_f\right)\boldsymbol{I} + V_f \boldsymbol{A}\right)^{-1} \tag{5}$$

where, $\boldsymbol{C_m}$ and $\boldsymbol{C}_r$ are the matrix and reinforcement stiffness tensor, respectively. Moreover, $\boldsymbol{I}$ is the fourth-order identity tensor. In the above equations a colon between two tensors denotes contraction (inner product) over two indices.

- Laminated analogy approach

The laminate analogy (LA) is a method which was commonly used to calculate the stiffness and strength of random short-fiber composites. This method was proposed by Halpin and Pagano [16]. In this method, it was assumed that the in-plane random material was treated as a stack of infinitesimally thin unidirectional plies bonded together with different fiber angle orientations. In this approach, the layers are interactive and the state of the stress in each layer affected by other layers properties. Chen [21] used the LA to calculate the stiffness and strength of composites. Halpin et al. [22] extended the LA to estimate the stiffness of the woven fabric composites. It was also shown that LA is able to calculate the elastic modulus of hybrid composites [23,24]. To the best knowledge of the present authors, the LA method has not been employed to estimate the mechanical properties of nanocomposites.

In the present article, the LA approach is employed to calculate the stiffness of nanocomposites. First, the LA approach is extended to a three dimensional form called 3-D LA. Then, using the stiffness matrix of the aligned reinforcement composites (obtained by the Mori-Tanaka method), the elastic modulus of the nanocomposites is calculated by the 3-D LA.

The random orientation of reinforcements in composites is one of the big challenges in modeling of this kind of materials. The first step of the modeling is to make the equivalent laminated composites (ELC) with appropriate simplifications and assumptions. The ELC is a laminated composite equivalent to the randomly-oriented composites. Then, using the classical lamination theory (CLT) the elastic modulus of the ELC is obtained. The reinforcements with similar orientation are considered to be located in one layer of