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A two-dimensional Riemann solver with self-similar sub-structure – Alternative formulation based on least squares projection

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ABSTRACT

Just as the quality of a one-dimensional approximate Riemann solver is improved by the inclusion of internal sub-structure, the quality of a multidimensional Riemann solver is also similarly improved. Such multidimensional Riemann problems arise when multiple states come together at the vertex of a mesh. The interaction of the resulting one-dimensional Riemann problems gives rise to a strongly-interacting state. We wish to endow this strongly-interacting state with physically-motivated sub-structure. The self-similar formulation of Balsara [16] proves especially useful for this purpose. While that work is based on a Galerkin projection, in this paper we present an analogous self-similar formulation that is based on a different interpretation. In the present formulation, we interpret the shock jumps at the boundary of the strongly-interacting state quite literally. The enforcement of the shock jump conditions is done with a least squares projection (Vides, Nkonga and Audit [67]). With that interpretation, we again show that the multidimensional Riemann solver can be endowed with sub-structure. However, we find that the most efficient implementation arises when we use a flux vector splitting and a least squares projection. An alternative formulation that is based on the full characteristic matrices is also presented. The multidimensional Riemann solvers that are demonstrated here use one-dimensional HLLC Riemann solvers as building blocks.

Several stringent test problems drawn from hydrodynamics and MHD are presented to show that the method works. Results from structured and unstructured meshes demonstrate the versatility of our method. The reader is also invited to watch a video introduction to multidimensional Riemann solvers on http://www.nd.edu/~dbalsara/Numerical-PDE-Course.

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1. Introduction

Riemann solvers play an important role in the numerical solution of hyperbolic systems of conservation laws. The one-dimensional Riemann problem is a self-similar solution that results from a discontinuity between two constant states. Multidimensional Riemann solvers have also been designed and we focus on a certain class of multidimensional Riemann solvers here (Wendroff [68], Balsara [4,5,16], Balsara, Dumbser and Abgrall [15], Vides, Nkonga and Audit [67], Balsara and Dumbser [17]). Such Riemann solvers are applied at the vertices of a two-dimensional mesh. Many states come together at a vertex from different directions, making it possible to communicate the multidimensionality of the flow to the multidimensional Riemann solver. At the vertex, the job of the multidimensional Riemann solver is to approximate the self-similar multidimensional structure that emanates from the vertex. While self-similarity has not been used much in the design of one-dimensional Riemann solvers, it is crucially important in the development of multidimensional Riemann solvers (Balsara [16], Balsara and Dumbser [17]). This has prompted the name of MuSIC Riemann solvers, where MuSIC stands for "Multidimensional, Self-similar, strongly-Interacting, Consistent". Such Riemann solvers are multidimensional; they draw on the self-similarity of the problem; they focus on the strongly-interacting state that results when multiple one-dimensional Riemann solvers interact; and the design relies on establishing consistency with the conservation law. MuSIC Riemann solvers that rely on a Galerkin projection to obtain the self-similar variation in the strongly interacting state have been presented (Balsara [16], Balsara and Dumbser [17]). An alternative projection method consists of least squares and Vides, Nkonga and Audit [67] developed a multidimensional Riemann solver without sub-structure based on such a projection. The *goal* of this paper is to show that least squares projection can also be used to design a MuSIC Riemann that retains sub-structure.

Several excellent one-dimensional Riemann solvers have been designed. There are exact Riemann solvers from Godunov [41,42] and van Leer [66] and two-shock approximations thereof (Colella [27], Colella and Woodward [29]). See also the work of Chorin [25]. The linearized Riemann solver by Roe [52] and the HLL/HLLE/HLLEM Riemann solvers (Harten, Lax and van Leer [44], Einfeldt [34], Einfeldt et al. [35]) and the local Lax–Friedrichs (LLF) Riemann solver (Rusanov [56]) have also seen frequent use. Toro, Spruce and Speares [62–64], Chakraborty and Toro [24] and Batten et al. [20] produced an HLLC class of Riemann solvers which have become very popular. See also, Billett and Toro [21]. Osher and Solomon [51] and Dumbser and Toro [33] presented approximate Riemann solvers based on path integral methods in phase space. In Balsara [16] we showed that the principle of self-similarity can be used to advantage with the result that any of the above-mentioned one-dimensional Riemann solvers can be used as a building block in the design of multidimensional Riemann solvers by relying on a Galerkin projection. The present paper continues this line of inquiry by showing that a least squares projection can also be used. The results are instantiated for the very popular HLLC class of Riemann solvers.

Magnetohydrodynamics (MHD) is an interesting example of a hyperbolic system with a more complex wave foliation. One-dimensional linearized Riemann solvers for numerical MHD have been designed (Roe and Balsara [54], Cargo and Gallice [23], Balsara [6]). HLLC Riemann solvers, capable of capturing mesh-aligned contact discontinuities, have been presented by Gurski [43] and Li [47]. Miyoshi and Kusano [49] drew on Gurski's work to design an HLLD Riemann solver for MHD. It is, therefore, interesting to show that MHD can also be accommodated within our formulation. MHD is a system with an involution constraint, where the divergence of the magnetic field is always zero. Balsara and Spicer [7] showed that this is assured within the context of a higher order Godunov scheme by using the upwinded fluxes at the edges of the mesh to update the magnetic fields that are collocated at the faces of a mesh. Gardiner and Stone [38,39] have claimed that the dissipation in those upwinded fluxes needs to be doubled all the time in order to stabilize the method. A substantial body of work now exists to show that the suggestion of Gardiner and Stone is completely unnecessary when multidimensional Riemann solvers are used to provide a properly upwinded electric field at the edges of the mesh (Balsara [5], Vides, Nkonga and Audit [67], Balsara and Dumbser [18]). Indiscriminate doubling of the dissipation, as per Gardiner and Stone's suggestion, can indeed lead to excessive dissipation of the magnetic field in the direction that is transverse to the upwind direction. The present paper reinforces that finding.

For the sake of completeness, and also for the sake of putting this work in context, we mention that there have been prior efforts at designing multidimensional Riemann solvers. One strain of research consists of trying to build some level of multidimensionality into one dimensional Riemann solvers (Colella [28], Saltzman [57], LeVeque [46]). Another line of early effort tried to incorporate genuine multidimensionality and did not meet with much initial success (Roe [53], Rumsey, van Leer and Roe [55]). Abgrall [1,2] made a big breakthrough by formulating a genuinely multidimensional Riemann solver for CFD that worked. Further advances were also reported (Fey [36,37], Gilquin, Laurens and Rosier [40], Brio, Zakharian and Webb [22], Lukacsova-Medvidova et al. [48]). Most of these above-mentioned genuinely multidimensional Riemann solvers did not see much use because they were difficult to implement. Wendroff [68] formulated a two-dimensional HLL Riemann solver, but his method was also not easy to implement. A video introduction to multidimensional Riemann solvers is available on the following website: http://www.nd.edu/~dbalsara/Numerical-PDE-Course.

Balsara [4] devised a two-dimensional HLL Riemann solver with simple closed form expressions for the fluxes that were easy to implement. In Balsara [5] it was shown that one can impart sub-structure to the HLL state, yielding a multidimensional HLLC Riemann solver. Balsara, Dumbser and Abgrall [15] extended this formulation to accommodate unstructured meshes. The previous three papers formulated the multidimensional Riemann problem by integrating the conservation law over the extent of the wave model in space–time. In their study of the multidimensional Riemann problem, Schulz-Rinne, Collins and Glaz [58] had shown that the one-dimensional Riemann problems interact amongst themselves to form a self-similarly evolving strongly-interacting state. This strongly-interacting state emerges by propagating into the one-dimensional