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Short note

# A third-order multistep time discretization for a Chebyshev tau spectral method

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ABSTRACT

A time discretization scheme based on the third-order backward difference formula has been embedded into a Chebyshev tau spectral method for the Navier–Stokes equations. The time discretization is a variant of the second-order backward scheme proposed by Krasnov et al. (2008) [3]. High-resolution direct numerical simulations of turbulent incompressible channel flow have been performed to compare the backward scheme to the Runge–Kutta scheme proposed by Spalart et al. (1991) [2]. It is shown that the Runge–Kutta scheme leads to a poor convergence of some third-order spatial derivatives in the direct vicinity of the wall, derivatives that represent the diffusion of wall-tangential vorticity. The convergence at the wall is shown to be significantly improved if the backward scheme is applied.



## 1. Introduction

In recent direct numerical simulations of turbulent channel flow with a spectral Chebyshev tau method, we noticed large errors in the budgets of the turbulence dissipation rate and enstrophy in the direct vicinity of the wall [1]. The time discretization scheme used was the Runge–Kutta method proposed by Spalart et al. [2], which is third-order accurate for the convective and second-order accurate for the viscous terms. Fortunately, these errors did apparently not affect the accuracy of quantities outside the direct vicinity of the wall, nor did they affect any mean value or standard deviation of a quantity based on the velocity or on spatial velocity derivatives up to order two. Nonetheless, the occurrence of these errors is disconcerting for a method which is regarded as one of the most accurate for direct numerical simulation of turbulent channel flow. The purpose of this note is to investigate this issue in more detail. It will appear that the issue is not resolved by spatial refinement if the time discretization method is not changed. We will therefore formulate and test an alternative time discretization method, which has at least the same formal order of accuracy as the Runge–Kutta method mentioned. It is a third-order variant of the second-order backward method proposed by Krasnov et al. [3].

## 2. A third-order time discretization scheme

The Chebyshev tau spectral method considered is based on the equation of the wall-normal component of the vorticity and the equation of the Laplacian of the wall-normal component of the velocity and has frequently been used in direct

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numerical simulation (DNS) of turbulent channel flow [4–8,3,9–13,1]. Only the wall-normal direction is treated with the Chebyshev tau discretization; the periodic streamwise and spanwise directions are treated with the Fourier spectral discretization [18,19]. Although the pressure has been eliminated from the equations, this Chebyshev tau method requires, like pressure-based Chebyshev methods [14–17], a type of influence matrix (Green's functions) technique for the enforcement of the correct wall boundary conditions.

In the first DNS of turbulent channel flow [4], this Chebyshev tau method was used in combination with the second-order accurate Crank–Nicolson Adams–Bashforth time discretization method (CN–AB). Later on, the Runge–Kutta method proposed by Spalart et al. [2], referred to as RK, was often used in spectral DNS of turbulent channel flow, see for example Refs. [5–8,11–13,1]. Compared to CN–AB, the advantages of RK are that it is self-starting and third-order accurate for the convective terms. Krasnov et al. [3] and others [9,10] combined the (pressureless) Chebyshev tau method with a time discretization scheme based on the second-order backward finite difference formula. However, these references do not clarify why this scheme was used instead of CN–AB or RK, which is more accurate for the convective terms. Since we prefer that an alternative to RK is at least third-order accurate for convective terms, we will introduce a third-order variant of the backward scheme of Krasnov et al. [3] in the following. We call this scheme B3.

The Chebyshev tau method considered is based on the evolution equations for four primary variables, two three-dimensional variables and two one-dimensional variables. The two three-dimensional primary variables are the wall-normal component of the vorticity ($\omega_2$) and the Laplacian of the wall-normal component of the velocity ($\phi = \Delta u_2$). The two one-dimensional primary variables are $U_1$ and $U_3$, the streamwise and spanwise velocity components averaged over the streamwise and spanwise directions. The third-order backward discretization of the four evolution equations can be written in the form,

$$\frac{11\hat{g}^{n+1} - 18\hat{g}^{n} + 9\hat{g}^{n-1} - 2\hat{g}^{n-2}}{6\,\delta t} = \nu \hat{L}\hat{g}^{n+1} + \hat{F}_g\left(\hat{\omega}_2^*, \hat{\phi}^*, \hat{U}_1^*, \hat{U}_3^*\right), \tag{1}$$

where the time-level is denoted in superscript and $\delta t$ is the time-step. The symbol $\hat{g}$ stands for the spectral transforms of $\omega_2$, $\phi$, $U_1$ and $U_2$. In addition, $\hat{L}$ denotes the spectral representation of the Laplace operator, $\nu$ the kinematic viscosity, and $\hat{F}_g$ denotes the nonlinear operator that expresses the non-viscous terms into the primary variables. The asterisk superscript indicates that the variable is extrapolated to time level $n+1$ using the third-order extrapolation formula,

$$\hat{q}^* = 3\hat{q}^n - 3\hat{q}^{n-1} + \hat{q}^{n-2}. \tag{2}$$

This completes the definition of time discretization scheme B3, which is formally third-order accurate for both convective and viscous terms. Apart from the order of accuracy, there are two other notable differences between scheme B3 and the second-order backward method of Krasnov et al. [3]. Whereas in Ref. [3] the linear extrapolation operator (2) is applied to nonlinear terms computed from non-extrapolated primary variables, in Eq. (1) the nonlinear operator is applied to extrapolated primary variables. The advantage of the latter is that no storage of nonlinear terms for reuse in future time steps is required. The primary variables do need to be stored, but this is required anyway, due to the left-hand side of Eq. (1). The second difference concerns the Von Neumann stability regime. Like scheme CN–AB, the second-order backward scheme is unstable in the inviscid limit, which means that the scheme is unstable on the entire imaginary axis, parametrized with $iy$ ($y$ is real), except if $y = 0$. However, scheme B3 is stable in the inviscid limit, since the stable region includes the interval $|y| < 0.63$. For comparison, the stable region of scheme RK includes the interval $|y| < 1.73$, almost three times larger, but the method is a three-stage method and therefore per time step three times as expensive as scheme B3. Due to the implicit treatment of the viscous terms, scheme B3 contains, like schemes CN–AB and RK, the entire left-half of the real axis.

Scheme RK is a low-storage scheme; the primary variables need only to be stored for the actual stage and the previous stage. In contrast, for scheme B3 the primary variables need to be stored for the actual time step and three previous time steps. Thus for the storage of primary variables scheme B3 requires twice as much memory as scheme RK. In our implementations, in which auxiliary variables are used for efficient computation of the dealiased nonlinear terms (based on $\mathbf{u} \times \boldsymbol{\omega}$), the total storage requirements for scheme B3 compared to scheme RK is much less higher than a factor of two.

## 3. Results

In the remainder of this note, we show how RK and B3 perform in high-resolution DNS of turbulent channel flow at $Re_\tau = 180$ (the channel half-width $H$ and the friction velocity $u_\tau$ are both equal to 1 and the kinematic viscosity $\nu$ is equal to 1/180). The flow is driven by a fixed mean streamwise pressure gradient. The streamwise, wall-normal and spanwise directions correspond to $x_1$, $x_2$ and $x_3$. These directions are discretized with $N_1$ Fourier modes, $N_2 + 1$ Chebyshev modes and $N_3$ Fourier modes, respectively. Nonlinear products are computed in physical space, with use of dealiasing in the $x_1$ and $x_3$ directions only (the grid in physical space contains $3N_1/2 \times (N_2 + 1) \times 3N_3/2$ points). The domain size is $L \times 2 \times L/3$. Statistics of the statistically steady state that will be shown have been obtained in the standard domain, $L = 4\pi$. The transient behaviour of the solution was studied in a smaller domain ($L = 2\pi$). The smaller domain size made it much more efficient to explore a large number of discretizations, grid sizes and error quantities.

In the discussion of the results, we focus on the vorticity equation. The $i$-component of the vorticity equation is given by,

$$\partial_t \omega_i = -\mathbf{u} \cdot \nabla \omega_i + \boldsymbol{\omega} \cdot \nabla u_i + \nu \Delta \omega_i, \tag{3}$$