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Residual-driven online generalized multiscale finite element methods

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ABSTRACT

The construction of local reduced-order models via multiscale basis functions has been an area of active research. In this paper, we propose online multiscale basis functions which are constructed using the offline space and the current residual. Online multiscale basis functions are constructed adaptively in some selected regions based on our error indicators. We derive an error estimator which shows that one needs to have an offline space with certain properties to guarantee that additional online multiscale basis function will decrease the error. This error decrease is independent of physical parameters, such as the contrast and multiple scales in the problem. The offline spaces are constructed using Generalized Multiscale Finite Element Methods (GMsFEM). We show that if one chooses a sufficient number of offline basis functions, one can guarantee that additional online multiscale basis functions will reduce the error independent of contrast. We note that the construction of online basis functions is motivated by the fact that the offline space construction does not take into account distant effects. Using the residual information, we can incorporate the distant information provided the offline approximation satisfies certain properties.

In the paper, theoretical and numerical results are presented. Our numerical results show that if the offline space is sufficiently large (in terms of the dimension) such that the coarse space contains all multiscale spectral basis functions that correspond to small eigenvalues, then the error reduction by adding online multiscale basis function is independent of the contrast. We discuss various ways computing online multiscale basis functions which include a use of small dimensional offline spaces.

1. Introduction

Solving real-world multiscale problems requires some type of model reduction due to disparity of scales. Many methods have been developed which can be classified as global [31,36,27] and local model reduction techniques [15,38,4,6,1,16,3,30,2,34,33,29,21,23,24,27,7,12,8,9,28]. Global model reduction techniques use global basis functions to construct reduced dimensional approximations for the solution space. These methods can involve costly offline constructions and lack local

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adaptivity. In this paper, our focus is on the development of efficient local multiscale model reduction techniques that involve some local online computations.

Many local multiscale model reduction techniques have been developed previously. These approaches solve the underlying fine-scale problems on a coarse grid. Among these approaches are upscaling techniques [15,38] and multiscale methods [6,16,21,23,24,27,7,12,8,9]. In the latter, multiscale basis functions are locally constructed that capture local information. Many research papers [25,37,22] have been dedicated to optimizing limited number of multiscale basis functions to capture the solution accurately. In some recent works [18,12,9,19,20], the authors develop Generalized Multiscale Finite Element method (GMsFEM). GMsFEM is a flexible general framework that generalizes the Multiscale Finite Element Method (MsFEM) [32] by systematically enriching the coarse spaces. The main idea of this enrichment is to add extra basis functions that are needed to reduce the error substantially. This approach, as in many multiscale model reduction techniques, divides the computation into two stages: the offline and the online. In the offline stage, a small dimensional space is constructed that can be used in the online stage to construct multiscale basis functions. These multiscale basis functions can be re-used for any input parameter to solve the problem on a coarse grid. The main idea behind the construction of offline and online spaces is the selection of local spectral problems and the selection of the snapshot space.

In subsequent papers [10,13], an adaptive GMsFEM is proposed. In these papers, we study an adaptive enrichment procedure and derive an a-posteriori error indicator which gives an estimate of the local error over coarse grid regions. The error indicators based on the L^2-norm of the local residual and on the weighted H^{-1}-norm of the local residual, where the weight is related to the coefficient of the elliptic equation are developed. We have shown that the use of weighted H^{-1}-norm residual gives a more robust error indicator which works well for cases with high contrast media. The error indicators contain the eigenvalue structure associated with GMsFEM. In particular, the smallest eigenvalue whose corresponding eigenvector is not included in the space enters into the error indicators.

Adaptivity is important for local multiscale methods as it identifies regions with large errors. However, after adding some initial basis functions, one needs to take into account some global information as the distant effects can be important. In this paper, we discuss the development of online basis functions that substantially accelerate the convergence of GMsFEM. The online basis functions are constructed based on a residual and motivated by the analysis.

We show, both theoretically and numerically, that one needs to have a sufficient number of initial basis functions in the offline space to guarantee an error decay independent of the contrast. We define such spaces as having online error reduction property (ONERP) and show that the eigenvalue that the corresponding eigenvector is not included in the offline space controls the error decay of the multiscale method. Larger is this eigenvalue, larger is the decrease in the error. Consequently, one needs to guarantee that eigenvectors associated with small (asymptotically small) eigenvalues are included in the initial coarse space. As we have discussed in [17,26], many multiscale problems with high contrast can have very small eigenvalues and, thus, we need to include the eigenvectors associated with small eigenvalues in the initial coarse space.

Numerical results are presented to demonstrate that one needs to have a sufficient number of initial basis functions in the offline space before constructing online multiscale basis functions. Moreover, we study how different dimensional offline spaces can affect the error decay when online multiscale basis functions are added. We consider several examples where we vary the dimension of the offline space and add multiscale basis functions based on the residual. Our numerical results show that without sufficient number of offline basis functions, the error decay is not substantial. We study the proposed online basis construction in conjunction with adaptivity [10,13], where online basis functions are added in some selected regions. Indeed, adaptivity is an important step to obtain an overall efficient local multiscale model reduction as it is essential to reduce the cost of online multiscale basis computations. Our numerical results show that the adaptive addition of online basis functions substantially improves GMsFEM. To reduce the computational cost associated with online multiscale basis computations, we propose computing the online basis functions in a reduced dimensional space consisting of several consequent offline basis functions. Our results show that one can still achieve a substantial error reduction this way. Because the online multiscale basis functions are not sparse in the offline space, approaches based on sparsity is not very helpful in our methods as our numerical results show.

In conclusion, the paper is organized in the following way. In Section 2, we present the underlying problem, the concepts of coarse and fine grids, and GMsFEM. In Section 3, we present some existing results for adaptive GMsFEM. In Section 4, we present our new proposed method for computing online multiscale basis functions. Numerical results are presented in Section 5. In Section 6, conclusions are drawn.

2. GMsFEM for high contrast flow

2.1. Overview

In this section, we will present a brief outline of the GMsFEM [18,12,9,19,20]. Let D be the computational domain. The high-contrast flow problem considered in this paper is

$$-\operatorname{div}\big(\kappa(x)\,\nabla u\big) = f \quad \text{in} \quad D, \tag{1}$$

with the homogeneous Dirichlet boundary condition $u = 0$ on ∂D, where f is a given source function. The difficulty in the numerical approximation of problem (1) arises from the complexity of the coefficient $\kappa(x)$, which can have multiple scales and very high contrast. In particular, discretizing (1) by traditional numerical schemes based on finite element or