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Staggered discontinuous Galerkin methods for the incompressible Navier–Stokes equations

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ABSTRACT

In this paper, we present a staggered discontinuous Galerkin method for the approximation of the incompressible Navier–Stokes equations. Our new method combines the advantages of discontinuous Galerkin methods and staggered meshes, and results in many good properties, namely local and global conservations, optimal convergence and superconvergence through the use of a local postprocessing technique. Another key feature is that our method provides a skew-symmetric discretization of the convection term, with the aim of giving a better conservation property compared with existing discretizations. We will present extensive numerical results, including Kovasznay flow, Taylor vortex flow, lid-driven cavity flow, parallel plate flow and channel expansion flow, to show the performance of the method.

1. Introduction

The paper concerns with numerical approximations of the incompressible Navier–Stokes equations by a class of discontinuous Galerkin (DG) methods. The DG method has been widely used for fluid flow and other related problems with great success, see for example [6,18,20–22,26,30,33,35,31,34]. On the other hand, many works in literature show that the use of staggered meshes is an important technique in computational fluid dynamics to reduce numerical dissipation [3,4,25], and in computational wave propagation to reduce numerical dispersion [8,9,12–15]. Therefore, it is natural to combine the ideas of DG and staggered mesh to develop a more convincing numerical scheme for computational fluid dynamics. In particular, we will develop a class of staggered discontinuous Galerkin (SDG) methods for the approximations of the incompressible Navier–Stokes equations. We will use an iteration approach. In each iteration, an Oseen problem, with the divergence-free convection velocity coming from the previous iteration, is solved by the SDG method.

One key component in the SDG discretization of the Oseen equation is a novel splitting of the diffusion and the convection term (cf. [17]). This splitting, together with the SDG discretization, results in a skew-symmetric discretization of the convection term. Similar to the results in [17], our new method also satisfies some local and global conservation properties. Another key component in the construction of our SDG method is that pointwise divergence-free velocity field is required

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to perform the above discretization. In general, the SDG method will only give weakly divergence-free velocity fields. To obtain a pointwise divergence-free velocity, we will use the postprocessing technique proposed in [18,19]. In addition to the above features, our SDG method has optimal rate of convergence, and the post-processed velocity is superconvergent. We remark that our SDG method is highly related to the HDG method [31] and the LDG method [20]. In particular, all the three methods are based on the first order formulation and a fixed point iteration using a postprocessed velocity field from the previous iteration. For the HDG and the LDG methods, the approximations are based on piecewise polynomials which are coupled by some numerical fluxes. For our SDG method, the approximation space is constructed using piecewise polynomials with some partial continuity conditions, and no numerical flux is necessary. We remark that our SDG method can be seen as the limit of some single face HDG methods, see [10,11]. One main advantage of our method is that the discretization of the convection term is skew-symmetric, which mimics better the continuous convection term. Such a discretization is spectro-consistent, which preserves the stability of the continuous convection operator and has better energy conservation as well as stability properties [36,37], see also Section 3.5.

Next, we introduce some notations. We consider the incompressible Navier–Stokes equations in a two-dimensional domain $\Omega \subset \mathbb{R}^2$:

$$
\begin{aligned}
-\boldsymbol{\Delta u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p &= \boldsymbol{f} \;\text{ in } \Omega,\\
\operatorname{div} \boldsymbol{u} &= 0 \;\text{ in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{u}_D \;\text{ on } \Gamma_D,\\
\nabla \boldsymbol{u} \cdot \boldsymbol{n} - p\boldsymbol{n} &= \boldsymbol{g}_N \;\text{ on } \Gamma_N,
\end{aligned} \tag{1}
$$

where p is the pressure, $\boldsymbol{u} = (u_1, u_2)$ is the velocity, $\boldsymbol{f} = (f_1, f_2)$ is the given source term and $\partial\Omega$ consists of disjoint parts Γ_D, Γ_N. We assume that $\int_\Omega p\, dx = 0$. By using an iteration procedure, we will consider the discretization of the Oseen problem:

$$
\begin{aligned}
-\boldsymbol{\Delta u} + \boldsymbol{V} \cdot \nabla \boldsymbol{u} + \nabla p &= \boldsymbol{f} \;\text{ in } \Omega,\\
\operatorname{div} \boldsymbol{u} &= 0 \;\text{ in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{u}_D \;\text{ on } \Gamma_D,\\
\nabla \boldsymbol{u} \cdot \boldsymbol{n} - p\boldsymbol{n} &= \boldsymbol{g}_N \;\text{ on } \Gamma_N,
\end{aligned} \tag{2}
$$

where $\boldsymbol{V}$ is a given divergence-free velocity field. For our SDG method, we introduce the auxiliary variables

$$
\boldsymbol{w} = \nabla u_1 - \frac{1}{2} u_1 \boldsymbol{V}, \quad \boldsymbol{z} = \nabla u_2 - \frac{1}{2} u_2 \boldsymbol{V}, \tag{3}
$$

and

$$
\widetilde{\boldsymbol{w}} = u_1 \boldsymbol{V}, \quad \widetilde{\boldsymbol{z}} = u_2 \boldsymbol{V}. \tag{4}
$$

Then problem (2) can be reformulated as

$$
\begin{aligned}
-\operatorname{div} \boldsymbol{w} + \frac{1}{2}\boldsymbol{V} \cdot \boldsymbol{w} + \frac{1}{4}\boldsymbol{V} \cdot \widetilde{\boldsymbol{w}} + p_x &= f_1 \;\text{ in } \Omega,\\
-\operatorname{div} \boldsymbol{z} + \frac{1}{2}\boldsymbol{V} \cdot \boldsymbol{z} + \frac{1}{4}\boldsymbol{V} \cdot \widetilde{\boldsymbol{z}} + p_y &= f_2 \;\text{ in } \Omega,\\
\operatorname{div} \boldsymbol{u} &= 0 \;\text{ in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{u}_D \;\text{ on } \Gamma_D,\\
\nabla \boldsymbol{u} \cdot \boldsymbol{n} - p\boldsymbol{n} &= \boldsymbol{g}_N \;\text{ on } \Gamma_N,
\end{aligned} \tag{5}
$$

together with the constraint that $\int_\Omega p\, dx = 0$. The SDG method in this paper is based on the first order system of (3), (4) and (5). We remark that, similar to [17], the resulting method has a skew-symmetric discretization of the convection term. Moreover, the discrete version of the additional variables $\boldsymbol{w}, \boldsymbol{z}, \widetilde{\boldsymbol{w}}$ and $\widetilde{\boldsymbol{z}}$ can be fully eliminated efficiently, and only the discrete velocity and pressure are computed.

The paper is organized as follows. In Section 2, we will give the construction of the SDG method. This includes the discretization of the Oseen problem and its application to the incompressible Navier–Stokes equations. Then, in Section 3, we will present extensive numerical examples to show the performance of the SDG method. Finally, a conclusion is given.

2. The SDG method

In this section, we will give the detailed construction of our SDG method. We will first construct the method for the Oseen equation (2), and then use it to solve the Navier–Stokes equations (1). To begin, we will give the construction of the staggered mesh, and the construction of finite element spaces with staggered continuity property. Next, we will explain the derivation of the SDG method. In addition, we will present the postprocessing technique (cf. [18]) and the application of our SDG method to the incompressible Navier–Stokes equations (1).