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On the monotonic and conservative transport on overset/Yin–Yang grids

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ABSTRACT

In this paper, we outline a simple and a general methodology to achieve positivity, monotonicity and mass conservation with transport schemes on general overset grids. The main feature of the approach is its reduced complexity, which simplifies the use of higher-order schemes and higher dimensions on general grids and in particular for overset grids. The method also does not degrade substantially the order of the overall scheme despite the extra constraints of monotonicity and conservation. The approach is applied to achieve mass conservation with semi-Lagrangian schemes and its performance is analyzed using simple one-dimensional overlapping grids and a two-dimensional spherical Yin–Yang grid. The Yin–Yang grid is a special overset grid for the sphere and it is of a special interest in the atmospheric modeling community, as it is one of the grids that may resolve the scaling issue of existing longitude–latitude-grid based atmospheric models on massively parallel machines.

1. Introduction

Numerical modeling, for many fields of science and engineering, consists of solving discrete analogues of the continuous partial differential equations, governing the physical problem, on a grid system. Depending on the problem and the geometry, the grids could be structured or un-structured. When the geometry permits, structured rectangular grids are the obvious choice due to the simplicity of their data structure and their suitability for finite differences and finite volumes methods.

Overset grids [4,21–23] are a technique whereby the domain of interest is divided into a number of overlapping subdomains and each sub-region has it own grid. The main reason for using these grids is to extend the use of basic rectangular grids and finite differences/volumes methods to complex geometries. The main attraction of these overset grids (also referred to in the literature as composite/overlapping/*Chimera* [22] mesh/grid) is the ability to handle a problem with a complex geometry using simple (in general rectangular) overlapping grids. Since the various sub-grids overlap each other, the main issue with such a technique is the treatment of overlap regions. These types of grids have been successfully applied in many fields of science and engineering, such as aerodynamics, hydrodynamics, combustion and electromagnetics [9].

For obvious reasons, atmospheric modeling has been dominated by Longitude–Latitude Grid (LLG). However, it is well known that the LLG suffers from a particular problem which is the convergence of meridians near the poles. This convergence of the meridians in the vicinity of the poles and the associated shrinkage of grid spacing has a direct effect on the computational performance of the model. For instance, the ratio between the physical mesh size at the equator and

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that near the poles has a direct impact on the condition number (stiffness) of the Helmholtz problem, resulting from a semi-implicit strategy. Finding a way of avoiding, or at least reducing the impact of, the singularity of the poles on LLG is as old as atmospheric modeling itself. Early approaches were the use of conformal projections, which preserve the angle between intersecting lines. For example, Phillips [15] used two stereographic projections for the polar regions coupled with a Mercator projection up to a certain latitude. Since then, various approaches have been proposed to resolve this issue, ranging from geodesic to overset grids [8,13,20,24].

For the sphere, the Yin–Yang grid [11] is an overset grid that divides the sphere into two identical rectangular (longitude-latitude) grids without the singularity of the poles. Despite many desirable properties, the Yin–Yang grid still has some issues related to the overlap region, which need further research as far as atmospheric modeling is concerned. This paper deals with one of such issues, which is how to achieve, efficiently, mass conservation with these grids.

Solving transport problems on overset and Yin–Yang grids have been dealt with using flux based schemes [5,14,25]. The method used in [5] computes as part of the solution modified interpolation weights for the overlap points to satisfy the conservation constraint. This however involves the solution of extra large system of equations for these weights, especially for higher dimensions and complex overlapping grids. The approach described in [14,25] relies on calculating detailed fluxes through the boundaries of the overlap regions and enforcing cancellation of in and out fluxes through the boundaries. This however requires the detailed knowledge of the overlapping grids geometry and complex book-keeping exercise to achieve mass conservation, which makes these schemes very complex, especially for higher-dimensions and when higher-order fluxes are required. In contrast, this paper describes a simple approach, with much reduced complexity, which can be applied straightforwardly to any general overset-grid, irrespective of the complexity of the geometry or the order of scheme required. The resulting scheme can achieve high-order solutions with the desired properties of monotonicity, positivity, mass conservation and computational efficiency in a straightforward manner, irrespective of the complexity of the grids or the dimension of the problem. The approach is successfully tested and validated on the Yin–Yang grid, due its special interest for the atmospheric community.

2. A general approach for positivity, monotonicity and conservation

Here we are concerned with the following general transport equation in either its conservative form:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho u) = 0, \tag{1}$$

or its non-conservative/advective form:

$$\frac{D\rho}{Dt} + \rho \nabla.u = 0, \tag{2}$$

where ρ is density; t is time; u velocity; $\nabla.()$ is the gradient operator and $D()/Dt \equiv \partial()/\partial t + u\nabla.()$ is the material (or Lagrangian) derivative.

2.1. One-dimensional problems

Suppose we can obtain, using any method, two valid solutions ρ^H and ρ^L, where ρ^H is some high-order oscillatory solution and ρ^L is a lower-order diffusive one. In the same spirit as classical FCT (Flux Corrected Transport) schemes [3], the main idea here is to seek a monotone (non-oscillatory) solution ρ^m using a linear combination of these two solutions:

$$\rho_i^m = \alpha_i \rho_i^H + (1-\alpha_i)\,\rho_i^L, \tag{3}$$

where $0 \le \alpha_i \le 1$ are local monotone-weights, and generally $\alpha_i \approx 0$ for unsmooth regions and $\alpha_i \approx 1$ for the smooth parts of the solution. Let $\widetilde{\rho}$ also be a monotone and conservative solution that lies between ρ^m and ρ^L:

$$\widetilde{\rho}_i = \beta_i \rho_i^m + (1-\beta_i)\,\rho_i^L, \tag{4}$$

where β_i are conservative weights which are as close as possible to unity (i.e., the aim is to have a final solution which is monotone and conservative but as close as possible to the high-order solution). Equations (3) and (4) can be combined to give:

$$\widetilde{\rho}_i = \beta_i \alpha_i \rho_i^H + (1-\beta_i\alpha_i)\,\rho_i^L. \tag{5}$$

The main problem now is to define the weights $(\alpha_i, \beta_i) \approx (1, 1)$ so that $\widetilde{\rho}_i \approx \rho_i^H$ while achieving both monotonicity and global mass conservation. In a similar fashion to the minmod type filter, the monotonicity weights α_i are given by:

$$\alpha_i = 1 - \max\left(\frac{(s_1 - |s_1|)\min(s_2, s_3)}{2|s_1 \min(s_2, s_3)|}, \frac{(|s_b| - s_b)}{2|s_b|}, 0\right), \tag{6}$$