Journal of Computational Physics 302 (2015) 674–692

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

A class of embedded discontinuous Galerkin methods for computational fluid dynamics

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ARTICLE INFO

Article history:
Received 24 August 2014
Received in revised form 9 June 2015
Accepted 12 September 2015
Available online 25 September 2015

Keywords:
Finite element methods
Discontinuous Galerkin methods
Hybrid/mixed methods
Euler equations
Navier–Stokes equations
Computational fluid dynamics

ABSTRACT

We present a class of embedded discontinuous Galerkin (EDG) methods for numerically solving the Euler equations and the Navier–Stokes equations. The essential ingredients are a local Galerkin projection of the underlying governing equations at the element level onto spaces of polynomials of degree k to parametrize the numerical solution in terms of the approximate trace, a judicious choice of the numerical flux to provide stability and consistency, and a global jump condition that weakly enforces the single-valuedness of the numerical flux to arrive at a global formulation in terms of the numerical trace. The EDG methods are thus obtained from the hybridizable discontinuous Galerkin (HDG) method by requiring the approximate trace to belong to smaller approximation spaces than the one in the HDG method. In the EDG methods, the numerical trace is taken to be continuous on a suitable collection of faces, thus resulting in an even smaller number of globally coupled degrees of freedom than in the HDG method. On the other hand, the EDG methods are no longer locally conservative. In the framework of convection–diffusion problems, this lack of local conservativity is reflected in the fact that the EDG methods do not provide the optimal convergence of the approximate gradient or the superconvergence for the scalar variable for diffusion-dominated problems as the HDG method does. However, since the HDG method does not display these properties in the convection-dominated regime, the EDG method becomes a reasonable alternative since it produces smaller algebraic systems than the HDG method. In fact, the resulting stiffness matrix has a similar sparsity pattern as that of the statically condensed continuous Galerkin (CG) method. The main advantage of the EDG methods is that they are generally more stable and robust than the CG method for solving convection-dominated problems. Numerical results are presented to illustrate the performance of the EDG methods. They confirm that, even though the EDG methods are not locally conservative, they are a viable alternative to the HDG method in the convection-dominated regime.

1. Introduction

Discontinuous Galerkin (DG) methods [3–5,19–23,28,29,31,34,35,38–41,57–59,61,64] have emerged as a competitive alternative for solving nonlinear hyperbolic systems of conservation laws because they possess some important advantages

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[1] This work was supported by AFOSR Grant No. FA9550-11-1-0141, AFOSR Grant No. FA9550-12-0357, and the Singapore-MIT Alliance for Research and Technology Centre.

http://dx.doi.org/10.1016/j.jcp.2015.09.024
0021-9991/

over classical finite differences and finite volume methods. In particular, they can easily handle complicated geometries, have low dissipation, are locally conservative, high-order accurate, highly parallelizable, and more robust than continuous Galerkin (CG) methods for convection-dominated problems. However, in spite of these advantages, DG methods have not yet made a more significant impact for practical applications. This is largely due to the main criticism that DG methods are computationally expensive since they have too many global degrees of freedom. Indeed, the high computational cost and memory storage are a major impediment to the widespread application of DG methods for real-world problems. Therefore, it would be highly desirable to develop new DG methods that have all the advantages of DG methods and are computationally competitive with continuous Galerkin (CG) methods and finite volume methods.

In the spirit of making DG methods more competitive, researchers have developed more efficient DG methods such as the multiscale discontinuous Galerkin (MDG) method [37,7] and the embedded discontinuous Galerkin (EDG) method [32,14]. The MDG method was originally introduced in the framework of convection–diffusion problems [37], whereas the EDG in the framework of linear shells [32] and linear diffusion [14]. These two methods, which can give rise to identical schemes, are devised to solve for a globally continuous approximation of the solution (the numerical trace of the scalar variable for the MDG, and the numerical trace of the approximate displacement for the EDG) on the element boundaries. A brief comparison of the ideas upon which these methods are defined is given in [32].

The EDG methods are constructed by using a suitable modification of an associated method called a hybridizable discontinuous Galerkin (HDG) method. The modification is applied to the variational formulation defining the system of globally-coupled degrees of freedom and consists in reducing its size by simply using a strict subspace to define it. The HDG method was introduced in [14] in the framework of diffusion problems. It was analyzed in [11,16,18,24,25] where it is shown that it has many common features with the Raviart–Thomas (RT) mixed method [60] and the Brezzi–Douglas–Marini (BDM) mixed method [6]. In particular, in [16], it was proven that the HDG method using simplexes and polynomials of degree $k \geq 0$ in all the unknowns, provides approximations to the scalar variable and the flux which converge with the optimal order $k+1$ in the L^2-norm for any $k \geq 0$, and that the element-by-element averages of the scalar variable superconverges with order $k+2$ for $k \geq 1$; a local postprocessing scheme can then be used to obtain a new approximation to the scalar variable converging with order $k+2$ for $k \geq 1$. The EDG method constructed from this HDG method by using *continuous* approximate traces for the scalar variable was analyzed in [17]. Therein, it was shown that, although this results in a smaller system for the globally-coupled degrees of freedom, it also produces the loss of the local conservativity of the method. As a direct consequence, although the approximation for the scalar variable still converged with order $k+1$ for any $k \geq 0$, the approximation for the flux converges with the *suboptimal* order of k. Hence, the above-mentioned postprocessing converges only with order $k+1$ for $k \geq 1$. Numerical experiments indicated that the EDG method was in fact *less* efficient than its associated HDG method.

This disappointing result precluded further study of EDG methods associated to HDG methods with similar optimal convergence and superconvergence properties. This is the case for HDG methods for linear convection–diffusion problems [51,12,8,9] and nonlinear convection–diffusion problems [12,52,65] in the diffusion-dominated regime, for HDG methods for the Stokes system of incompressible flow [13,15,47,53,26], for HDG methods for the incompressible Navier–Stokes equations [49,50,54] in the diffusion-dominated regime, and for HDG methods for linear elasticity [26,27]. In particular, a unique feature of the HDG method for incompressible fluid flow is that the approximate velocity, pressure and velocity gradient converge with the optimal order $k+1$ in the L^2-norm for diffusion-dominated flows for any $k \geq 0$. Moreover, the element-by-element averages of the velocity superconverge and, a local postprocessing scheme proposed in [15,49] can be used to obtain a new approximate velocity which converges with order $k+2$ for $k \geq 1$.

On the other hand, it is reasonable to believe that, in the convection-dominated regime, all the above-mentioned HDG methods must behave like the classic DG methods in the pure convection limit. As a consequence, the optimality of the convergence, for example, to the gradient of the scalar variable, in the case of convection–diffusion problems is lost along with the above-mentioned superconvergence property. Numerical evidence of this fact is provided in [12]. It is then reasonable to expect that in this situation, the EDG methods might prove to be more efficient than the original HDG method. In other words, the EDG method [55] constructed from the HDG method for the compressible Euler and Navier–Stokes equations [44–46,56] in the convection-dominated regime, might be more efficient than the HDG method.

In this paper, we extend our previous work [55] to develop a class of EDG methods. The EDG methods are obtained from the HDG method by requiring the approximate trace to belong to smaller spaces than the one in the HDG method. In the EDG methods, the numerical trace is taken to be continuous on a suitable collection of faces, thus resulting in an even smaller number of globally coupled degrees of freedom than in the HDG method. Indeed, the original EDG method [55] produces a global matrix system that has the same sparsity pattern as that of the statically condensed continuous Galerkin (CG) method. As one instance of the class of EDG methods developed in this paper, the interior embedded DG (IEDG) method has slightly less globally coupled unknowns than the EDG method. Furthermore, the IEDG method enforces the boundary conditions more accurately than the EDG method. Numerical results presented herein confirm that the IEDG method outperforms the EDG method and thus establishes itself as a viable alternative to the HDG method.

The paper is organized as follows. In Section 2, we introduce the notation used throughout the paper and compare various DG methods in terms of the number of degrees of freedom and the number of nonzeros in their Jacobian matrix. We then introduce the class of EDG methods for the Euler equations in Section 3 and extend it to the compressible Navier–Stokes equations in Section 4. In Sections 3 and 4, we present numerical results to demonstrate the performance of the EDG methods. Finally, in Section 5, we end the paper with some concluding remarks.