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Noise-induced transition in barotropic flow over topography and application to Kuroshio

Wenqi Yao [a], Weiqing Ren [a,b,*]

[a] Department of Mathematics, National University of Singapore, 119076, Singapore
[b] Institute of High Performance Computing, Agency for Science, Technology and Research, 138632, Singapore

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ABSTRACT

The minimum action method, developed by E et al. (2004) [1] based on the least action principle from the Wentzell–Freidlin theory of large deviations, is applied to barotropic flow over topography. Application is presented for Kuroshio. The optimal dynamical paths for transitions between the small and large meander states are obtained by minimizing the action functional with the initial and final states being constrained in the metastable sets. The action is minimized using the BFGS quasi-Newton method and the constraints are imposed using the penalty method with a barrier function. It is found that the transition from the small meander state to the large meander state proceeds by nucleation and subsequent propagation of small eddies near the corner of the Kyushu peninsula. The transition backwards follows roughly the same path though details of the flow are different. The transition states are identified by examining the action along the paths.

1. Introduction

In this paper, we explore the phase space of barotropic flows over topography and compute the most likelihood transition pathways using the minimum action method [1]. The minimum action method was developed for the study of rare events. Rare events are the events that happen infrequently compared with the relaxation time scale of the system. Usually there is a small amount of noise in the system. The system stays at metastable states for most of time. However, due to the presence of the noise, the system may hop between metastable states or make excursions out of these states over long time scales. When such an event happens, it usually happens very quickly and has important consequences. Chemical reactions, conformational changes of bio-molecules, laminar-to-turbulence transitions in fluids, nucleation events in phase transitions are all examples of such rare events.

Quantitative understanding of the effect of noise and the associated transition events has attracted a lot of attention in recent years. When the noise is small, the dynamics contains two disparate time scales: the time scale of deterministic dynamics and the time scale between the transition events caused by the noise. In this case, traditional methods, such as the Monte Carlo method or direct simulation of the Langevin equation, become prohibitively expensive.

Noting this difficulty, alternative methods have been developed. The most notable analytical work is the Wentzell–Freidlin theory of large deviations [2,3], which gives an estimate on the probability of the paths in the terms of an action functional. The most probable transition path is given by the minimum action path (MAP), i.e. the path that minimizes the action

* Corresponding author at: Department of Mathematics, National University of Singapore, 119076, Singapore.
E-mail addresses: matyw@nus.edu.sg (W. Yao), matrw@nus.edu.sg (W. Ren).

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functional. For gradient systems with an underlying potential or free energy landscape, the MAP over an infinite time interval coincides with the minimum energy path (MEP), which minimizes the energy barrier along the path. The MEP has a very simple characterization: It is a heteroclinic orbit connecting the minima of the energy and it goes through the saddle point. For non-gradient systems, which is the focus of this paper, such a simple characterization does not exist, and the MAP may be much more complex.

Various numerical methods for the study of rare events have also been proposed. Most notable among these methods are the nudged elastic band method [4] and the string method [5,6] for finding MEPs as well as the transition rates. The Wentzell–Freidlin theory of large deviations was exploited as a numerical tool by E et al. [1], and the minimum action method (MAM) was developed for spatially extended dynamical systems and applied in many fields, such as [7,8] etc. In this method, the MAP is computed by minimizing the action functional with the initial and final states being constrained at the metastable states. The minimum action method was later extended in different directions. In Ref. [9], the action functional was reformulated and minimized in the space of geometric curves parameterized by arc length. In Refs. [10,11], an adaptive mesh method was used in the parameterization of the path, which aims at producing a better resolution of the path in the transition state region.

In this paper, we study noise-induced transition events in barotropic flow over topography using the minimum action method. The flow may exhibit metastable behavior due to complex structures of the topography. The Kuroshio is such an example. The Kuroshio refers to the ocean current to the south of Japan. It exhibits bimodality, i.e. the small meander state and the large meander state. In the small meander state, the ocean current stays close to the coastline, while in the large meander state, the current follows a path which meanders a large distance into the ocean in the region between the Kyushu peninsula and the Izu ridge [12]. Each of th two metastable state may persist for a long period (from 3 to 10 years) before switching to the other [13]. In this paper, we apply the MAM to compute the most likely transition path, i.e. the MAP, between the small and large meander states.

We model the flow using the barotropic vorticity equation (BVE), with an appropriate topography function characterizing the sea bed between the Kyushu peninsula and the Izu ridge. This system becomes increasingly non-stationary even in the deterministic case as the mass transport increases. As a result, the small and large meander states actually correspond to two metastable sets, whose sizes depend on the magnitude of mass transport. For a prescribed transition time, the action and the MAP are quite sensitive to the choice of initial and final states within these sets. To overcome this difficulty, instead of fixing the initial and final states of the path at two prescribed states when minimizing the action, we constrain the initial and final states in the metastable sets of the small and large meander states respectively. As a result, the initial and final states of the MAP are part of the solution. They are computed together with the MAP by solving the constrained minimization problem. In the computation, the constraints by the metastable sets are imposed using a penalty method.

The paper is organized as follows. In Section 2, we introduce the vorticity equation for barotropic flow over topography and the model for Kuroshio. We also discuss the bistable behavior of Kuroshio and present the small and large meander states obtained from numerical solution of the barotropic vorticity equation. In Section 3, we introduce the minimum action method and apply the least action principle to transition events between the small and large meander states of Kuroshio. In Section 4, we discuss the discretization of the action functional and the algorithm for solving the constrained minimization problem. Numerical results are presented in Section 5. Some conclusions are drawn in Section 6.

2. Barotropic flow over topography

The BVE model introduced in [12] is used to model the dynamics of Kuroshio

$$\frac{\partial \xi}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\xi \boldsymbol{u}) + \nabla_{\boldsymbol{x}} f \cdot \boldsymbol{u} - f \nabla_{\boldsymbol{x}} (\ln h) \cdot \boldsymbol{u} = \nu \nabla^2_{\boldsymbol{x}} \xi, \tag{1}$$

where $\boldsymbol{x} = (x, y) \in D$, $t \in \mathbb{R}^+$, ν is the kinematic viscosity, $h(x, y)$ models the topography at the bottom of the physical domain, and f is the linearized Coriolis force

$$f = f_0 + f_x x + f_y y, \tag{2}$$

where $(f_x, f_y) = (\beta \sin\theta, \beta \cos\theta)$, with β being the conventional Rossby parameter and θ being the angle between the x-axis and the equator of the earth. $\xi = \xi(t, x, y)$ is the vorticity and $\boldsymbol{u} = (u, v)$ is the depth-averaged velocity on the (x, y) plane. Denote the stream function of the flow by $\psi = \psi(t, x, y)$. Then the velocity $\boldsymbol{u}$ and the vorticity ξ can be written as

$$(u, v) = \left(-\frac{1}{h}\frac{\partial \psi}{\partial y}, \frac{1}{h}\frac{\partial \psi}{\partial x} \right), \tag{3}$$

and

$$\xi = \frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial \psi}{\partial y}\right). \tag{4}$$