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A high order approximation of hyperbolic conservation laws in networks: Application to one-dimensional blood flow

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ABSTRACT

We present a methodology for the high order approximation of hyperbolic conservation laws in networks by using the Dumbser–Enaux–Toro solver and exact solvers for the *classical* Riemann problem at junctions. The proposed strategy can be applied to any hyperbolic system, conservative or non-conservative, and possibly with flux functions containing discontinuous parameters, as long as an exact or approximate Riemann problem solver is available. The methodology is implemented for a one-dimensional blood flow model that considers discontinuous variations of mechanical and geometrical properties of vessels. The achievement of formal order of accuracy, as well as the robustness of the resulting numerical scheme, is verified through the simulation of both, academic tests and physiological flows.

1. Introduction

We are concerned with hyperbolic conservation laws for which relevant applications include the treatment of networks consisting of one-dimensional domains, hereafter referred to as *edges*, sharing their boundaries, hereafter referred to as *vertexes*. Relevant applications regard traffic flow [1], gas flow in pipes [2,3], open channel flows and water distribution networks [4–6] and the modeling of the circulatory system [7–11]. A crucial aspect for this kind of applications is how to provide boundary conditions at terminal vertexes and algebraic coupling conditions between one-dimensional domains at internal vertexes. A common practice for choosing the coupling conditions is to identify a problem specific quantity, as could be the pressure in the case of isentropic Euler equations [12], or the total pressure in the case of blood flow [13]. Then, the numerical scheme is adapted in order to consistently enforce the devised coupling condition.

In the case of explicit schemes that make use of numerical fluxes at computational cell interfaces, such as finite volume or discontinuous Galerkin schemes, there exists a formal approach that can be pursued. In practice, defining the coupling conditions at an internal vertex P, shared by N_P edges, can be seen as the resolution of a *classical* Riemann Problem (RP) for a junction of N_P one-dimensional domains. This problem was deeply discussed in [6], for example. Under the assumption of sub-sonic flow, the hyperbolic system will provide the necessary wave relations to link the states provided by one-dimensional domains sharing a vertex, as in the case of the classic Riemann problem [14]. Therefore, in order to obtain coupling conditions that are completely consistent with the underlying hyperbolic conservation law, the Riemann

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problem at the junction has to be solved and the resulting Godunov state can then be used to compute numerical fluxes needed by the explicit scheme to evolve the solution within the one-dimensional domain.

In the case of explicit high order finite volume schemes, one has non-constant data as initial conditions for the Riemann problem at the junction. This specific Cauchy problem is known as the Generalized Riemann Problem (GRP) [15]. In the case of an internal vertex, it is essential to adopt a numerical methodology that solves the GRP at this location accurately enough in order to preserve the formal accuracy of the scheme used for the resolution of conservation laws within the one-dimensional domain. To the best of our knowledge, the first work presenting a methodology to achieve arbitrary high order accuracy in this context was proposed in [16], where the authors used a particular version of a class of numerical schemes known as ADER schemes [17], allowing for arbitrary high order accuracy via one-step numerical methods.

Here, always within the ADER framework, we adopt a different methodology, the Dumbser–Enaux–Toro Riemann solver [18] and adapt certain ingredients to the specificity of the problem under study. This GRP solver requires a high order spatial reconstruction, which in this work is performed using the WENO procedure [19,20] and a classical RP solver. Special attention must be placed in the choice of the classical RP solver, since in this kind of applications parameters present in the flux function might be discontinuous and the RP solver should be able to deal with this characteristic of the problem in an appropriate manner.

The proposed methodology allows for arbitrarily accurate approximations of hyperbolic conservation laws in networks. In order to illustrate and validate the methodology, we implement the high order treatment of coupling conditions for a one-dimensional blood flow model, using the exact classical RP solver proposed in [21]. This mathematical model incorporates several specific features, such as the treatment of discontinuous parameters in the flux function and the propagation of shocks (or elastic jumps). These features can be found in other applications, such as gas flow in pipes with varying cross-sectional area [22] or open channel flow with variable topography [23].

The rest of the paper is structured as follows. In Section 2 we introduce the mathematical model chosen to illustrate the proposed methodology, which is then developed. Next, in Section 3 we perform a series of tests to verify the accuracy and robustness of the proposed methodology. First we verify that the expected order of accuracy is achieved by solving a test with exact solution. Then we solve a series of test problems on simple networks, including smooth and discontinuous solutions. We conclude the tests by solving the one-dimensional equations on an arterial network model with appropriate boundary conditions. Final remarks are drawn in Section 4.

2. Methods

In this section we present the mathematical model adopted to illustrate the proposed methodology. Then we show how to solve the *classical* RP at an internal vertex. Finally, we propose how to accurately solve the GRP at this location.

2.1. One-dimensional blood flow model with variable vessel properties

One-dimensional blood flow in deformable vessels is described by the following system

$$\begin{cases} \partial_t A + \partial_x q = 0\,, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{A}\right) + \dfrac{A}{\rho}\partial_x p = -\dfrac{f}{\rho}\,, \end{cases} \tag{1}$$

where x is the axial coordinate along the longitudinal axis of the vessel; t is time; $A(x,t)$ is the cross-sectional area of the vessel; $q(x,t)$ is the flow rate; $p(x,t)$ is the average internal pressure over a cross-section; $f(x,t) = \gamma\pi\mu\frac{q}{A}$ is the friction force per unit length of the tube, with γ depending on the velocity profile; μ is the fluid viscosity and ρ is the fluid density.

Pressure $p(x,t)$ is related to the cross-sectional area $A(x,t)$ by the algebraic relation of the form

$$p(x,t) = K(x)\,\phi(A(x,t), A_0(x)) + p_e(x,t) = K(x)\left[\left(\frac{A(x,t)}{A_0(x)}\right)^m - \left(\frac{A(x,t)}{A_0(x)}\right)^n\right] + p_e(x,t)\,. \tag{2}$$

Here, $p_e(x,t)$ is a given external pressure, and $K(x)$, m, n, $A_0(x)$ are parameters that take into account mechanical and geometrical properties of the vessel.

Equations (1) must be complemented with appropriate boundary conditions at terminal vertexes of the network and so-called coupling conditions at internal vertexes or junctions. Concerning junctions, if we consider N_P vessels sharing a vertex, we must define N_P state vectors $\mathbf{Q}_*^k$ with $k = 1, \ldots, N_P$ (see (9) below), in order to provide coupling conditions for each one of the vessels converging to the vertex P. A coupling condition is mass conservation

$$\sum_{k=1}^{N_P} g_P^k q_*^k = 0\,, \tag{3}$$

where g_P^k is the auxiliary function

$$g_P^k = \begin{cases} 1\,, & \text{if } x_P^k = L^k\,, \\ -1\,, & \text{if } x_P^k = 0 \end{cases} \tag{4}$$