Journal of Computational Physics 300 (2015) 455–491

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Improved second-order hyperbolic residual-distribution scheme and its extension to third-order on arbitrary triangular grids

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ARTICLE INFO

Article history:
Received 26 January 2015
Received in revised form 16 July 2015
Accepted 18 July 2015
Available online 31 July 2015

Keywords:
Higher-order
SUPG
Fluctuation splitting
Finite volume
Navier–Stokes
Quadratic reconstruction
Hermite polynomial
High-order integration

ABSTRACT

In this paper, we construct second- and third-order hyperbolic residual-distribution schemes for general advection–diffusion problems on arbitrary triangular grids. We demonstrate that the accuracy of the second-order hyperbolic schemes in [J. Comput. Phys. 227 (2007) 315–352] and [J. Comput. Phys. 229 (2010) 3989–4016] can be greatly improved by requiring the scheme to preserve exact quadratic solutions. The improved second-order scheme can be easily extended to a third-order scheme by further requiring the exactness for cubic solutions. These schemes are constructed based on the SUPG methodology formulated in the framework of the residual-distribution method, and thus can be considered as economical and powerful alternatives to high-order finite-element methods. For both second- and third-order schemes, we construct a fully implicit solver by the exact residual Jacobian of the proposed second-order scheme, and demonstrate rapid convergence, typically with no more than 10–15 Newton iterations (and about 200–800 linear relaxations per Newton iteration), to reduce the residuals by ten orders of magnitude. We also demonstrate that these schemes can be constructed based on a separate treatment of the advective and diffusive terms, which paves the way for the construction of hyperbolic residual-distribution schemes for the compressible Navier–Stokes equations. Numerical results show that these schemes produce exceptionally accurate and smooth solution gradients on highly skewed and anisotropic triangular grids even for a curved boundary problem, without introducing curved elements. A quadratic reconstruction of the curved boundary normals and a high-order integration technique on curved boundaries are also provided in details.

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1. Introduction

In many flow simulations, accurate predictions of solution gradients, such as velocity and temperature gradients, are essential for design and analysis purposes as they are directly related to the physical quantities of interest: e.g., the viscous stresses, the vorticity, and the heat fluxes. However, it is widely accepted that accurate and smooth solution gradients cannot be achieved with conventional schemes on fully irregular unstructured grids [1,2]. In conventional schemes, the gradients are obtained typically with a lower order of accuracy (e.g., through reconstruction of primary variables), and they

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http://dx.doi.org/10.1016/j.jcp.2015.07.054
0021-9991/Published by Elsevier Inc.

are usually subject to numerical oscillations on such grids. The resolution of this issue is very important in justifying the use of high-fidelity models in engineering design, analysis, and optimization, especially for applications involving complex geometries. The ability to predict the gradients on irregular grids is even more critical for grid adaptation, a vital technique for efficient CFD calculations in high-order methods [3], because the grid adaptation almost necessarily introduces irregularity in the grid. In fact, current practices in grid adaptation often avoid adaptation in certain regions such as within boundary layers where grid irregularity has a severe impact on the solution quality [4]. Therefore, numerical schemes that can accurately predict solution gradients on irregular grids need to be developed, so that the power of grid adaptation can be fully exploited.

There have been some efforts in the residual-distribution community in developing schemes that provide high-order solutions [5–10], but little attention is paid, in general, in developing schemes that produce accurate and smooth high-order solution gradients. A good review of residual-distribution schemes is given in Ref. [11], in which a third-order residual-distribution scheme was presented for quadratic (P_2) elements. A different third-order residual-distribution scheme was also proposed and applied to RANS equations in Ref. [12], and for real-gas computations in Ref. [13]. These schemes produce no better than second-order accurate solution gradients, whether linear or quadratic elements are employed. A more improved third-order residual-distribution scheme was later proposed in Refs. [14,15], and third-order accurate solution gradients were reported on some unstructured grids, including hybrid elements, using a proposed special gradient reconstruction strategy. However, the authors have noted that the third-order accurate solution gradients could not be reproduced for randomly distorted triangular elements. Quality of the predicted solution gradients using these schemes were not reported and therefore, are unknown.

The first-order hyperbolic system method (or the hyperbolic method for short), with which the solution and the solution gradients are simultaneously computed by solving a hyperbolic system for diffusion, provides a platform for construction of high-order schemes that could potentially produce high-order solution gradients that are both accurate and free of numerical noise.

The hyperbolic method was first studied for diffusion in Ref. [16] and then for advection–diffusion in Ref. [17] with Residual-Distribution (RD) schemes. Later, the method was demonstrated for the compressible Navier–Stokes equations by a second-order Finite-volume (FV) scheme [18]. Since then, there have been efforts in developing high-order hyperbolic schemes in the finite–volume method [19–21], in the active flux method [22], and in the RD method [23,24] for unsteady computations.

In this paper, we focus on the development of hyperbolic RD schemes for two-dimensional problems, extending the previous work [16,17], with several important advances.

- Improved Accuracy: We propose to construct a second-order scheme that preserves exact quadratic solutions, which can be accomplished by the curvature correction technique [8]. The resulting scheme remains compact for viscous problems, and produces significantly improved solution gradients over the previous schemes, which do not preserve exact quadratic solutions.
- Third-Order Accuracy: Extending the improved second-order scheme, we construct a third-order scheme that preserves exact cubic solutions. The construction requires quadratic least-squares (LSQ) gradients and a high-order source term discretization developed here.
- Nonlinear Equation: The improved schemes are extended to a nonlinear advection–diffusion equation by the preconditioned conservative formulation introduced in Ref. [18].
- High-Order on Linear Elements: We demonstrate that the third-order scheme does not require curved elements for curved boundary problems; it gives more accurate solution and gradients than the second-order scheme on the same linear grids (straight-sided elements). Our scheme is designed to be third-order accurate on straight-sided triangles, even for geometries containing curved boundaries. This is a significant advantage, because most high-order methods require curved geometries to be represented by high-order curved elements; see Ref. [3]. Our proposed high-order scheme is not the only scheme that produces high-order solution on unstructured, straight-sided meshes. For example, the technique of Ref. [25], which was applied to Discontinuous Galerkin (DG) method, produces high-order accurate solution for geometries containing curved boundaries by locally approximating the curvature of the physical geometry (i.e., high-order normals) using information from the neighboring boundary elements (i.e., a local operation) with all triangles kept as straight-sided elements (not curved). The finite-volume (FV) scheme of Ref. [26] is another example, where a third-order solution was obtained on the linear elements with a quadratic reconstruction of the boundary normals for curved boundaries. The third-order residual-distribution schemes of Refs. [27,8,7], which are developed based on reconstruction techniques, are additional examples. Our proposed third-order scheme is among these schemes, and is more aligned with the FV scheme of Ref. [26] because the proposed third-order scheme here produces third-order solution gradients for geometries containing curved boundaries that are represented by straight-sided meshes.
- Non-Unified Approach: Instead of the fully integrated approach of discretizing the hyperbolic advection–diffusion system as in Ref. [17], we discretize the advective and diffusive terms separately. This approach will enable the extension to the compressible Navier–Stokes equations for which the eigenstructure of the whole system has not been discovered yet.
- Fully Implicit Solver: We construct a fully implicit solver for both second- and third-order schemes. For practical applications, explicit iterations considered in Refs. [16,17] are not efficient enough, and a fully implicit solver is needed. The