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Finite difference approximations of multidimensional convection–diffusion–reaction problems with small diffusion on a special grid

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ABSTRACT

A numerical scheme for the convection–diffusion–reaction (CDR) problems is studied herein. We propose a finite difference method on a special grid for solving CDR problems particularly designed to treat the most interesting case of small diffusion. We use the subgrid nodes in the Link-cutting bubble (LCB) strategy [5] to construct a numerical algorithm that can easily be extended to the higher dimensions. The method adapts very well to all regimes with continuous transitions from one regime to another. We also compare the performance of the present method with the Streamline-upwind Petrov–Galerkin (SUPG) and the Residual-Free Bubbles (RFB) methods on several benchmark problems. The numerical experiments confirm the good performance of the proposed method.

1. Introduction

It is well known that the exact solution of the convection–diffusion–reaction (CDR) problems may contain layers when some problem parameters are too big compared to others. Typically in this model problem, but also in real fluid flow simulation, the major difficulty is the appearance of the nonphysical oscillations that pollute the numerical solution in the whole domain, while the exact solution only shows boundary or internal layers. To overcome this difficulty, several numerical recipes have been evolved over the years [32,33] among them a commonly used one is the finite difference method [1,11,13,27]. The early numerical solutions were obtained by using standard finite difference scheme of upwind and centered type on a uniform mesh and then refining the mesh more and more in order to capture the boundary/internal layers. However, even for 1-D problems those methods were inefficient and accurate solutions could not be obtained for higher dimensions. In [3], Bakhvalov considered an upwind difference scheme on a layer-adapted meshes. They are very fine inside the boundary layer and coarse outside. Moreover, in 1990s the Russian mathematician Shishkin showed that one could use a simpler piecewise uniform mesh to obtain reasonable approximations [14,36]. This idea has been propagated throughout the 1990s by a group of Irish mathematicians: Miller, O'Riordan and Farrell [29]. The simplicity of those approaches is due to the use of equidistant subintervals on both sides of a transition point and this property is considered to be one of its major attractions. However, it requires the precise location of the layer structure a priori.

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Another major approach to obtain reasonable approximations for the CDR problem is the finite element method (FEM) [2,14,21]. The most successful classes of FEMs for treating convection-dominated problems are achieved by the stabilized formulations [16,18,19,22,24,37]. As an important and well-known example to that class, the Streamline-Upwind/Petrov–Galerkin (SUPG) method could be mentioned that is first proposed by Hughes and his co-workers [10]. SUPG method is based on enlarging the variational formulation by adding diffusion in the streamline direction while preserving the consistency. Despite the success of SUPG method, the need for the proper choice of stabilizing parameter is considered as a major drawback of the method. Regarding that fact, intrinsically stable methods such as the Residual-Free Bubbles (RFB) method has been developed [2,4,8,9,17]. The main idea underlying the RFB method is to enrich the finite element space, instead of a modification of the variational formulation, by a set of special functions, so called bubble functions. A thorough comparison of some of these methods can be found in [12,28,38]. However, it requires to solve a local differential equation which may not be easier than to solve the original one [15]. That observation has motivated the introduction of a further option so-called the Pseudo Residual-free Bubble (PRFB) method which approximates the bubble functions on a specially chosen subgrid [6,7,31,34,35]. Roughly speaking, such grid points can be constructed by minimizing the residual of a local differential equation with respect to L_1 norm so that small scale-effect of the exact solution could be accurately represented in the numerical approximation through the use of those approximate bubble functions [34]. Alternatively, the Link-Cutting Bubbles (LCB) method that is based on the plain Galerkin variational formulation on a special grid was proposed by Brezzi et al. in [5] and it could be viewed as a similar, yet interesting option for another stable discretization in 1D. However, extension of that strategy to the higher dimensions is not a trivial task. It is also worth mentioning that the convection–diffusion–reaction equations with positive and negative reactive terms (source terms) is considered in [23].

The algorithm investigated in this work is motivated by a simple splitting of the 2-D CDR equation into the sum of two 1-D equations [25]. It combines the ideas of the LCB method in [5] and finite difference methods (FDM) on special meshes. Indeed, we will use the subgrid nodes in the LCB strategy and construct a FDM for solving CDR problems. Thus, we develop a numerical recipe for solving CDR problems that is simple to use, easy to implement and can easily be extended to higher dimensions. We also compare the performance of the present method with the well-known SUPG and RFB methods on several benchmark problems. A wide range of problem parameters has been examined on both structured and unstructured meshes and the corresponding numerical results are presented.

The layout of the paper is as the following. We briefly recall the basic idea of the LCB method in Section 2. In Section 3, we describe the details of the numerical method proposed and discuss the generation of the grid for two dimensional problem. Finally, we perform the numerical tests for several benchmark problems in both 2D and 3D in Section 4.

2. A review of the Link-Cutting Bubble strategy in [5]

We consider the following linear elliptic convection–diffusion–reaction problem on a unit interval $I = (0, 1)$

$$\begin{cases} \mathcal{L}u = -\epsilon u'' + \beta u' + \sigma u = f(x) \text{ on } I, \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

under the assumptions that the diffusion coefficient ϵ is positive constant, convection field β and reaction field σ are non-negative constants. We denote the decomposition of I into subintervals by $T_h = \{K_k\}$ where $K_k = (x_{k-1}, x_k)$, $k = 1, \ldots, N$ and the size of the interval K_k by $h_k = x_k - x_{k-1}$.

The Link-Cutting Bubble (LCB) strategy introduced in [5] is designed for one-dimensional convection–diffusion–reaction problem and it aims to mimic the stabilizing effect of Residual Free Bubbles (RFB), without actually computing them. To do this, we choose a suitable subgrid made of two points inside each element and we take the bubbles which are piecewise linear on the subgrid. The strategy for choosing the subgrid is as follows: Consider a typical element, (x_1, x_2), then the subgrid nodes are obtained by adding two extra nodes, say z_1 and z_2 satisfying $x_1 < z_1 < z_2 < x_2$ and

$$\begin{aligned} z_1 - x_1 &= \min\left\{h_k - 2(x_2 - z_2), \frac{3\beta + \sqrt{9\beta^2 + 24\epsilon\sigma}}{2\sigma}\right\} \\ x_2 - z_2 &= \min\left\{\frac{h_k}{3}, \frac{-3\beta + \sqrt{9\beta^2 + 24\epsilon\sigma}}{2\sigma}\right\}. \end{aligned} \tag{2}$$

Once the subgrid nodes are constructed, the LCB strategy works as the standard Galerkin method with piecewise linear basis functions on augmented mesh. For the behavior of the scheme at various regimes, see [5].

3. The construction of the numerical method

In this section, using the subgrid nodes in the LCB strategy, we propose a numerical algorithm for solving convection–diffusion–reaction (CDR) problems which can easily be extended to the higher dimensional problems. Now, consider the following constant coefficient linear elliptic convection–diffusion–reaction problem in a polygonal domain Ω: