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Generalized adjoint consistent treatment of wall boundary conditions for compressible flows

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ABSTRACT

In this article, we revisit the adjoint consistency analysis of Discontinuous Galerkin discretizations of the compressible Euler and Navier–Stokes equations with application to the Reynolds-averaged Navier–Stokes and k–ω turbulence equations. Here, particular emphasis is laid on the discretization of wall boundary conditions. While previously only *one specific* combination of discretizations of wall boundary conditions and of aerodynamic force coefficients has been shown to give an adjoint consistent discretization, in this article we generalize this analysis and provide a discretization of the force coefficients for *any* consistent discretization of wall boundary conditions. Furthermore, we demonstrate that a related evaluation of the c_{p}- and c_{f}-distributions is required. The freedom gained in choosing the discretization of boundary conditions without loosing adjoint consistency is used to devise a new adjoint consistent discretization including numerical fluxes on the wall boundary which is more robust than the adjoint consistent discretization known up to now.

While this work is presented in the framework of Discontinuous Galerkin discretizations, the insight gained is also applicable to (and thus valuable for) other discretization schemes. In particular, the discretization of integral quantities, like the drag, lift and moment coefficients, as well as the discretization of local quantities at the wall like surface pressure and skin friction should follow as closely as possible the discretization of the flow equations and boundary conditions at the wall boundary.

1. Introduction

A discretization is adjoint consistent if the discrete adjoint problem is a consistent discretization of the continuous adjoint equations [1–3]. Adjoint consistency – in addition to consistency – is the key property of a discretization to be of optimal order in the L^2-norm as well as measured in terms of target functionals [4–6]. Furthermore, discrete adjoint solutions to adjoint consistent discretizations are smooth whereas they might be irregular for adjoint inconsistent discretizations [2,3, 6–8].

While the adjoint consistency analysis has originally been developed for Discontinuous Galerkin (DG) discretizations of the linear advection equation, of Poisson's equation and of compressible flow equations [1–3,6,7] it has been transferred and applied to a variety of other problems and/or other discretization schemes [9–17]. The general framework of the adjoint consistency analysis as provided in [3] can be employed to analyze adjoint consistency properties of discretizations as

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well as to identify discretization terms due to which a discretization is adjoint inconsistent. In that case, the analysis helps to find modifications to the discretization to make it adjoint consistent. Having a Discontinuous Galerkin discretization at hand which is already adjoint consistent on interior faces (e.g., SIPG and BR2 discretizations for viscous terms are adjoint consistent, whereas NIPG is not [1]) the most critical issue is the discretization at boundary faces in combination with the discretization of target quantities like aerodynamic force coefficients. First ignoring viscous terms, for the DG discretization of the compressible Euler equations, following combination of discretizations has been found in [2,3,7] to be adjoint consistent:

$$\hat{\mathbf{h}}_\Gamma(\mathbf{u}_h^+,\mathbf{n}) = \mathbf{n}\cdot\mathcal{F}^c(\mathbf{u}_\Gamma(\mathbf{u}_h^+)), \tag{1a}$$

$$J_h(\mathbf{u}_h^+) = J(\mathbf{u}_\Gamma(\mathbf{u}_h^+)). \tag{1b}$$

Here, $\mathbf{u}_h^+$ denotes the interior trace of the discrete state vector in conservative variables of an element near the wall boundary. As DG discretizations incorporate boundary conditions in a weak sense, $\mathbf{u}_h^+$ does in general not satisfy the slip wall (i.e., vanishing normal velocity) boundary condition. In contrast to that, $\mathbf{u}_\Gamma(\mathbf{u}_h^+)$ is computed from $\mathbf{u}_h^+$ by removing its normal velocity component and thus represents a *wall boundary state* with vanishing normal velocity. Furthermore, the particular example of a *numerical boundary flux* $\hat{\mathbf{h}}_\Gamma(\mathbf{u}_h^+,\mathbf{n})$ provided in (1a) is given by the normal component $\mathbf{n}\cdot\mathcal{F}^c$ of the convective flux of the compressible Euler equations evaluated at the wall boundary state $\mathbf{u}_\Gamma(\mathbf{u}_h^+)$, and $J_h(\mathbf{u}_h^+) = J(\mathbf{u}_\Gamma(\mathbf{u}_h^+))$ in (1b) represents a (consistent) discretization of the drag or lift coefficient given by $J(\mathbf{u}) = \int_{\Gamma_W} p(\mathbf{u})\,\mathbf{n}\cdot\psi\,\mathrm{d}s$ (cf. Section 2.2 for more details) again evaluated at the wall boundary state. Restricted to the compressible Euler equations the main outcome of [2,3] was that a discretization including the normal boundary flux (1a) is adjoint consistent if and only if the force coefficient is evaluated based on (1b). In contrast to that, (1a) in combination with $J(\mathbf{u}_h^+)$ would be adjoint *in*consistent.

$\hat{\mathbf{h}}_\Gamma$ in (1a) is one of many different possible choices of numerical boundary fluxes. In fact, for the consistency of the discretization at the boundary it is only required that $\hat{\mathbf{h}}_\Gamma(\mathbf{u}_h^+,\mathbf{n})$ is consistent, i.e., that it reduces to the normal convective flux, $\hat{\mathbf{h}}_\Gamma(\mathbf{u},\mathbf{n}) = \mathbf{n}\cdot\mathcal{F}^c(\mathbf{u})$, if evaluated for the exact solution $\mathbf{u}$. Besides (1a) another quite prominent choice of a numerical boundary flux is given by

$$\hat{\mathbf{h}}_\Gamma(\mathbf{u}_h^+,\mathbf{n}) = \hat{\mathbf{h}}(\mathbf{u}_h^+,\mathbf{u}_\Gamma^\star(\mathbf{u}_h^+),\mathbf{n}). \tag{2}$$

Here, $\hat{\mathbf{h}}$ represents any (consistent) numerical flux function connecting the interior state $\mathbf{u}_h^+$ and a derived state $\mathbf{u}_\Gamma^\star(\mathbf{u}_h^+) \neq \mathbf{u}_h^+$ on the wall boundary. Here, typically the same numerical flux (e.g., the Roe flux [18] or the local Lax–Friedrichs flux, among many others) is employed like on interior faces. In contrast to the normal boundary flux (1a) the numerical flux function involved in (2) introduces some numerical dissipation at the boundary. This typically leads to an increased stability and robustness of the discretization scheme. In fact, our experience from numerical experiments is that DG discretizations with (2) are in general more stable than with (1a). However, [2,3] showed that (2) in combination with any of the discretizations of the target quantity considered in [2,3] would be adjoint *in*consistent. Therefore, since [2,3] we were restricted to use either the adjoint consistent thus more accurate but possibly less robust discretization based on (1a), or the adjoint inconsistent, less accurate but more robust discretization based on (2).

In this article, we now generalize the adjoint consistency analysis and provide a discretization of the force coefficients for *any* consistent discretization of wall boundary conditions. This will in future offer the possibility to choose one of the discretizations based on (1a), or (2), or any other consistent discretization on the wall boundary and still obtain adjoint consistency if the discretization of the force coefficients is chosen accordingly. In particular, in Sections 2 and 3, we consider the compressible Euler and Navier–Stokes equations, together with the associated (continuous) adjoint equations. We derive DG discretizations of them in the most general form and analyze them with respect to consistency and adjoint consistency. In particular, here we introduce discretizations of the aerodynamic force coefficients which result in adjoint consistent discretizations for *any* consistent discretization of boundary conditions. Furthermore, in contrast to previous works, here the adjoint consistency analysis includes the treatment of farfield boundary conditions. Section 4 then extends the discretization to the Reynolds-averaged Navier–Stokes (RANS) and Wilcox k–ω turbulence equations.

Related to the evaluation (or discretization) of integral quantities like the force coefficients is the evaluation of local quantities at the wall boundary like surface pressure and skin friction as involved in the c_p- and c_f-distributions. The local c_p-value is given by $c_\mathrm{p}(\mathbf{u}) = \frac{p(\mathbf{u})-p_\infty}{0.5\rho_\infty v_\infty^2}$, where $p(\mathbf{u})$ denotes the pressure evaluated at the state $\mathbf{u}$, and p_∞, ρ_∞ and v_∞ denote farfield quantities of the pressure, density and velocity, respectively. As an example, the blue and red lines in Fig. 1 show the c_p-distributions evaluated based on $c_\mathrm{p}(\mathbf{u}_\Gamma(\mathbf{u}_h^+))$ for the L1T2 high-lift configuration (cf. Section 6.3 for more details), computed with a DG discretization based on the normal boundary flux in (1) and the numerical flux function (2). Due to the coarse resolution (coarse grid and low polynomial degree) both DG solutions show some discontinuities. However, most remarkable is the saw-tooth type c_p-distribution (in red) obtained with the discretization based on (2). While $c_\mathrm{p}(\mathbf{u}_\Gamma(\mathbf{u}_h^+))$ is connected to the adjoint consistent treatment of the force coefficient in (1b) and gives a relatively smooth c_p-distribution (in blue) in Fig. 1, it is, however, clearly inappropriate for the evaluation of the c_p-distribution for the discretization based on (2).

In this article, we extend the adjoint consistent treatment of the boundary conditions and force coefficients to an adjoint consistent evaluation of the c_p- and c_f-distributions. In particular, for any discretization of the wall boundary condition we