Journal of Computational Physics 300 (2015) 844–861

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

One-way spatial integration of hyperbolic equations

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ARTICLE INFO

Article history:
Received 17 February 2015
Received in revised form 17 July 2015
Accepted 10 August 2015
Available online 14 August 2015

Keywords:
Hyperbolic
One-way equation
Parabolic approximation
Parabolized
Spatial marching

ABSTRACT

In this paper, we develop and demonstrate a method for constructing well-posed one-way approximations of linear hyperbolic systems. We use a semi-discrete approach that allows the method to be applied to a wider class of problems than existing methods based on analytical factorization of idealized dispersion relations. After establishing the existence of an exact one-way equation for systems whose coefficients do not vary along the axis of integration, efficient approximations of the one-way operator are constructed by generalizing techniques previously used to create nonreflecting boundary conditions. When physically justified, the method can be applied to systems with slowly varying coefficients in the direction of integration. To demonstrate the accuracy and computational efficiency of the approach, the method is applied to model problems in acoustics and fluid dynamics via the linearized Euler equations; in particular we consider the scattering of sound waves from a vortex and the evolution of hydrodynamic wavepackets in a spatially evolving jet. The latter problem shows the potential of the method to offer a systematic, convergent alternative to ad hoc regularizations such as the parabolized stability equations.

1. Introduction

Many physical phenomena can be modeled by systems of linear hyperbolic partial differential equations. A defining characteristic of hyperbolic systems is that their solutions are comprised of waves that propagate at finite speeds. The well-posed solution of these equations on a finite computational domain takes the form of either an initial boundary value problem in the time domain or an elliptic boundary value problem if the equations are transformed into the frequency domain. In either case, incoming waves must be specified at the boundaries.

In some situations, the solution is dominated by waves that propagate in one direction. We call these waves rightgoing and the waves that propagate in the opposite direction leftgoing. Approximate one-way equations are often sought to represent the rightgoing waves. These one-way equations, also known as parabolized or parabolic equations, are valuable because they can be rapidly solved in the frequency domain by spatial integration in the direction of wave propagation. For the spatial march to be accurate and well-posed, the one-way equation must approximate the same rightgoing waves as the original hyperbolic equation but not support any leftgoing waves. If support for the leftgoing waves is not properly removed, decaying leftgoing waves are wrongly interpreted as growing downstream waves, causing instability in the spatial march.

One-way equations have been formally derived for various wave equations. Most one-way wave equations are derived by factoring the dispersion relation in Fourier–Laplace space. Two factors are obtained – one representing rightgoing waves

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http://dx.doi.org/10.1016/j.jcp.2015.08.015
0021-9991/

and one representing leftgoing waves. A one-way wave equation is obtained by only retaining the rightgoing branch. The resulting equation contains a square root involving the Fourier–Laplace variables, and the inverse transform of this term results in a nonlocal integro-differential equation. Computationally efficient one-way wave equations can be obtained by localizing the operator using rational approximations of the square root [1]. Early methods involved approximations at a fixed low order [2,3], while subsequent versions generalized this idea to arbitrarily high-order approximations. The accuracy and well-posedness of different approximations have been extensively studied [4,5]. One-way wave equations are routinely used to study geophysical migration [6,7] and underwater acoustics [8,9], and, when transformed back to the time domain, can be used as approximate nonreflecting boundary conditions [10,11].

These methods are very efficient and accurate for simple wave equations but degenerate quickly in more complicated situations because of their dependence on the factorization of the equations in Fourier–Laplace space, which, for example, is not possible for equations whose eigenvalues cannot be written down analytically. It is therefore not straightforward to apply these techniques to general hyperbolic systems. Guddati [12] developed a method for the acoustic and elastic wave equations that does not depend on this factorization. His method shares some qualitative similarities to the scheme we propose in this paper, but it has not been extended beyond wave equations.

The linearized Euler and Navier–Stokes equations can be spatially integrated using an ad hoc generalization of linear stability theory called the parabolized stability equations (PSE) [13]. PSE is designed to track the one-way spatial evolution of a single rightgoing wave, usually the most spatially amplified wave supported by the system. The wavelength and growth-rate of this wave are assumed to be slowly-varying. Instead of formally deriving a one-way operator, PSE achieves a stable spatial march by numerically damping all leftgoing waves, either by using an implicit axial discretization along with a restriction on the *minimum* step size [14] or by explicitly adding damping terms to the equations [15]. This damping prevents the leftgoing waves from destabilizing the spatial march, but also has the unintended consequence of damping and distorting, to differing degrees, all of the rightgoing waves.

PSE has been used extensively to study instability waves in slowly-spreading shear-flows. It is often the case that the near-field solutions of these equations are dominated by a single amplifying rightgoing wave related to classic instability modes of the Orr–Sommerfeld operator. This wave can be calculated very efficiently with reasonable accuracy using PSE. On the other hand, other rightgoing waves supported by the Euler equations, for example rightgoing acoustic waves, are not properly captured by PSE because of the aforementioned damping.

A number of other spatial marching methods have been developed for solving the Euler and Navier–Stokes equations that are collectively known as reduced or parabolized Navier–Stokes equations. After neglecting viscous terms in the parabolization direction, leftgoing acoustic waves are eliminated by special treatment of the streamwise pressure gradient. A number of variations exist in which this term is treated differently, ranging from neglecting it partially [16] or entirely [17] to prescribing it based on experimental data [17] or empirical approximations. Classical boundary layer equations fall into this category.

In this paper, we describe and demonstrate a new technique for developing accurate one-way approximations of linear hyperbolic systems. Our method formally removes support for leftgoing waves from the equations without analytically factorizing the dispersion relation, resulting in well-posed equations that can be solved by spatial marching without the need for numerical damping. As a result, the rightgoing waves can be accurately captured for systems in which the leftgoing waves are unimportant. In Section 2, we first derive exact one-way equations based on concepts related to the well-posedness of hyperbolic boundary value problems and then show how the exact equations can be efficiently approximated using techniques that were originally developed for generating high-order nonreflecting boundary conditions. The method is applied to the Euler equations in Section 3, and the accuracy and efficiency of the resulting one-way Euler equations is demonstrated in Section 4 using three example problems. Finally, we discuss possible improvements to the method and conclude the paper in Section 5.

2. Method

In this section, we formulate our parabolization method. First, we derive the spatial boundary value problem for a general hyperbolic system. Then, we derive an exact one-way equation for systems that are homogeneous along the axis of parabolization by identifying and eliminating leftgoing waves. The exact formulation is computationally expensive, so we next formulate efficient, well-posed approximations of the exact equations. We discuss the application of these methods to systems that vary along the axis of parabolization and finally compare the computational cost of the one-way equations to standard solution techniques.

2.1. Problem formulation

We begin with a system of linear, strongly hyperbolic partial differential equations

$$\frac{\partial q}{\partial t} + A\left(x, y\right)\frac{\partial q}{\partial x} + \sum_{j=1}^{d-1} B_j\left(x, y\right)\frac{\partial q}{\partial y_j} + C\left(x, y\right) q = 0. \tag{1}$$