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High-fidelity numerical simulation of the dynamic beam equation

Ken Mattsson *, Vidar Stiernström

Department of Information Technology, Uppsala University, P O Box 337, S-751 05 Uppsala, Sweden

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ABSTRACT

A high-fidelity finite difference approximation of the dynamic beam equation is derived. Different types of well-posed boundary conditions are analysed. The boundary closures are based on the summation-by-parts (SBP) framework and the boundary conditions are imposed using a penalty (SAT) technique, to guarantee linear stability. The resulting SBP–SAT approximation leads to fully explicit time integration. The accuracy and stability properties of the newly derived SBP–SAT approximations are demonstrated for both 1-D and 2-D problems.

1. Introduction

The dynamic beam equation (DBE), also known as the Euler–Bernoulli beam equation, is a standard model of flexible body dynamics and is thus of interest in many engineering applications where beams are used as the basis of supporting structure or as axles, e.g. when studying vibrations of buildings or railway structures. The DBE is derived from Euler–Bernoulli beam theory, one of the simplest beam theories dating back to the 18th century. The model includes potential energy arising due to strain forces from the bending of the beam and kinetic energy due to the lateral displacement of the beam. The governing equation of the DBE is a linear partial differential equation (PDE) that is second order in time and fourth order in space.

In the literature, numerical analysis of the DBE in time-domain is very scarce, in particular concerning higher-order methods. Most numerical methods target the analysis of natural frequencies of beams (see for example [4]), through Laplace transformation of the DBE and study a single frequency at a time. FEM is a well-proven technique for these types of time-independent problems, especially if non-cartesian geometries are considered. A drawback with this approach is that time-dependent phenomena cannot easily be approximated, as well as broadband sources. In the present study we analyse and solve the DBE in time-domain using an explicit finite difference method. This approach is computationally much more demanding (than solving the DBE in frequency domain) since the dispersive nature of the DBE requires very small time-steps to resolve the high-frequency part of the solution. A well-proven approach to reduce the computational cost (or rather the degrees of freedom) is to employ a high-order accurate finite-difference method. (See the pioneering paper by Kreiss and Oliger [18].) The major difficulty with higher-order finite difference methods is to obtain a stable boundary treatment, which has received considerable past attention. (For examples, see [20,38,36,1,5,13].)

* Corresponding author. Tel.: +46 18 4717631; fax: +46 18 523049.
E-mail address: ken.mattsson@it.uu.se (K. Mattsson).

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The summation-by-parts–simultaneous approximation term (SBP–SAT) method is a robust and well-proven high-order finite difference methodology that ensures stability of time-dependent PDEs. The SBP–SAT method combines semi-discrete operators that satisfy a summation-by-parts (SBP) formula [17], with physical boundary conditions (BC) implemented using the simultaneous approximation term (SAT) method [6]. (The SAT method was originally developed for pseudo-spectral approximations [8,9]. Examples of the pseudo-spectral penalty approach can be found in [13,42].) Examples of the SBP–SAT approach can be found in [34,26,29,30,35,24,19,7,15,14,16,22,2].

The SBP–SAT approach has so far been developed for problems involving first and second derivatives in space. However, there are many problems where higher order derivatives are present. Some examples include the Korteweg–de Vries and the Boussinesq equations (describing nonlinear water waves), soliton models in neuroscience [37], the Euler–Lagrange equation for beams [41,10], and the Cahn–Hilliard equation which describes the process of phase separation. Recently, high-order accurate SBP operators for third and fourth derivatives were derived [32].

In contrast to hyperbolic problems [24,25,40,3] solutions to the DBE are dispersive, meaning that the group velocity of a wave governed by the DBE is frequency dependent. Hence, the DBE is a dispersive wave equation. To capture the dispersive nature of the equation efficiently it is even more essential that high-order (i.e. higher than second order) spatially accurate numerical methods are used to capture the high-frequency parts of the solution. The time-dependent Schrödinger equation is also a dispersive wave equation that has been successfully solved using the SBP–SAT method [2,33]. However, the numerical treatment of the DBE is much more challenging due to the high order derivative. Another distinction is that the DBE has a second derivative in time while the Schrödinger equation has a first derivative in time. (By splitting the Schrödinger equation into real and imaginary components it is possible to obtain an equation similar to the DBE for the real component.)

Since the wave speed scales as the frequency, the physics require that we take much smaller time-step (k) compared to the spatial grid-size (h), in order to resolve the fastest going waves (i.e., the high-frequency part of the solution). Hence, the physics require $k \simeq h^2$. In the present study we will derive an explicit time-integration with a CFL condition similar to the physical time-step requirement. This physical restriction on the time-step is another motivation to employ spatially high-order accurate methods to allow for larger h. As the DBE involves a fourth derivative in space, the numerical boundary treatment is challenging. In particular the treatment of *clamped* BC, requires novel treatment when employing the SBP–SAT technique.

The main focus in the present study is to construct high-order accurate explicit (i.e., do not require solving any equation system to obtain the difference approximation) SBP–SAT approximations of the DBE, for quite general type of BC.

In Section 2 the DBE is introduced, including the most common types of BC. In Section 3 the SBP–SAT method is introduced. Stability analysis of the various SBP–SAT approximations is presented in Section 4. Time integration is analysed and discussed in Section 5. In Section 6 the accuracy and stability properties of the newly developed SBP–SAT approximations are verified by performing numerical simulations. The extension to 2-D applications is addressed in Section 7. Section 8 summarises the work. The SBP operator for the fourth-order accurate case is presented in Appendix A.

2. The dynamic beam equation

For a beam of length L with its axis along the x-direction, denote the deflection of the beam from its axis as $u(x,t)$. The governing equation of the 1-D DBE is then given by,

$$\begin{aligned} &\mu\frac{\partial^2 u(x,t)}{\partial t^2} = -\frac{\partial^2}{\partial x^2}\left(E(x)I(x)\frac{\partial^2 u(x,t)}{\partial x^2}\right) + F(x,t), \quad 0 \le x \le L,\ t \ge 0 \\ &u(x,0) = f_1(x), \qquad \frac{d}{dt}u(x,0) = f_2(x), \quad 0 \le x \le L, \end{aligned} \tag{1}$$

where $E(x)$ is the elastic modulus of the beam, $I(x)$ the second moment of area of the cross section of the beam and μ the mass per unit length. Here $f_{1,2}(x)$ are initial data and $F(x,t)$ a forcing function. For a homogeneous beam, E and I are independent of x.

Remark. The stability analysis in the present study allows for variable coefficients, but would require SBP operators that can approximate $(a(x)u_{xx})_{xx}$. Such operators do not yet exist. In the present study we assume constant coefficients since SBP operators for constant coefficient fourth derivatives were recently derived in [32]. There are many applications that have piecewise constant coefficients. Treatment of piecewise constant coefficients has been done earlier for the acoustic wave equation using constant coefficient SBP operators by introducing internal interfaces (see for example [27,24]). This extension is something we intend to analyse in a coming study.

Using the notation u_{xxxx}, u_t and u_{tt} for the fourth, first and second partial derivatives of $u(x,t)$ in space and time respectively, the DBE is reduced to,

$$\begin{aligned} &\mu u(x,t)_{tt} = -EIu(x,t)_{xxxx} + F(x,t), \quad 0 \le x \le L,\ t \ge 0 \\ &u(x,0) = f_1(x), \qquad u(x,0)_t = f_2(x), \quad 0 \le x \le L. \end{aligned} \tag{2}$$