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On vector field reconstructions for semi-Lagrangian transport methods on geodesic staggered grids

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ABSTRACT

We analyse several vector reconstruction methods, based on the knowledge of only specific pointwise vector components, and extend their use to non-structured polygonal C-grids on the sphere. The emphasis is on the reconstruction of the vector field at arbitrary locations on the sphere, as required by semi-Lagrangian transport schemes. This is done by first reconstructing the vector field to fixed locations, followed by interpolations with generalized barycentric coordinates. We derive a hybrid scheme, combining the efficiency of Perot's method with the accuracy of a least square scheme. This method is second order accurate, and has shown to be competitive and computationally efficient. We analysed the vector reconstruction methods within a semi-Lagrangian transport method, and demonstrated that second order accurate reconstructions are enough to fulfil the requirements for second order accurate semi-Lagrangian methods on icosahedral C-grids.

1. Introduction

Quasi-uniform geodesic grids have achieved considerable importance in global atmospheric modelling and have been adopted in some recent developed models (e.g. [1,2]). One of the motivations for this choice is that models based on quasi-uniform grids are prone to scale better on massively parallel computers than several operational models making use of latitude–longitude grids [3]. Nevertheless, interesting questions arise in the development of models on quasi-uniform grids, mainly related to their non-orthogonality and lack of structure. In the present work, we shall concentrate on hexagonal/pentagonal icosahedral geodesic C-grids, or generally, on Voronoi like C-grids (including spherical centroidal tessellations [4]). The use of staggering of variables, especially of the C-grid type, favours a better representation of fast waves [3]. The aspects to be emphasised in this work are related to the development of semi-Lagrangian models on icosahedral grids. Semi-Lagrangian schemes, either used as a transport method for moisture and other tracers, or combined with semi-implicit discretizations in dynamical cores, play an important role on many operational atmospheric models. A crucial part in a semi-Lagrangian scheme is an accurate trajectory computation. In the computation of the trajectories, semi-Lagrangian schemes usually require sufficiently good approximations to the wind field at any point on the sphere, based on the available wind information. Usually, only the normal components of the winds at the midpoints of the edges of the Voronoi tessellation will be known (in some schemes, the tangential components may be available instead). The problem of obtaining the wind

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field at an arbitrary location from the available wind components is often called the vector field reconstruction problem, focus of this work.

Vector field reconstructions have been considered by Wang et al [5] for triangular C-grids on the plane, in the context of modelling oceanic flows. They have examined the use of finite element based methods (as the Raviart–Thomas (RT) triangular elements [6]), of Perot's methods [7] and of polygonal least square (LSQ) reconstructions [8]. Bonaventura et al. [9] considered the use of radial basis functions (RBF) and compared them to RT reconstructions, showing that RBF based methods outperform the RT schemes on triangular grids. All these schemes have been originally developed for planar triangular grids, with the exception of RBF methods.

The purpose of this paper is to investigate vector reconstruction methods for general spherical polygonal C-grids. We aim to reconstruct the vector fields at arbitrary locations on the sphere, as required in usual semi-Lagrangian trajectory computations [10]. With this objective, we devise ways to extend edge-based finite element methods (such as RT and Whitney [11] elements), Perot's schemes and least square reconstructions to spherical polygonal C-grids and compare their accuracy and computational efficiency, also with the RBF methods.

As observed in [5] for triangular grids, nonuniformity of grid cells influence the accuracy of the methods. The grid properties may affect the precision of the schemes, to the point that interpolation and discretization errors exhibit a marked grid-pattern, what is known as grid-imprinting [12]. We observe this behaviour with the edge-based finite element methods and with Perot's method. In [13] we were able to directly relate the grid-imprinting of finite volume discretizations of the divergence operator to an alignment property (introduced in [13]) of the grid cells. With a similar analysis, we can explain the error patterns of Perot's method. Moreover, with the aid of the cell alignment indexes, we can devise a hybrid reconstruction method, combining the (computationally cheap) method of Perot on well aligned cells with a more expensive least square method, used only on the minority of badly aligned cells. The resulting scheme has similar accuracy to least square methods, at a much lower cost. This hybrid scheme also shows to be competitive to RBF methods using 6 or 9 point stencils, with the advantage of not requiring the precomputation and storage of matrices decompositions. As observed in [9], and well investigated in the literature [14], RBF methods present a certain duality between precision and numerical stability. For fixed size stencils, the condition numbers of the systems to be solved grow very fast for finer grids. For the resolutions in current use, the systems are still stable and RBF methods provide good approximations [1,2], but this may be a potential problem in the way to cloud resolving models. The addition of a polynomial part to the RBF basis may contribute to alleviate this problem. We investigated the use of Gaussian kernels with an extra constant term in the reconstruction of vector fields, with some interesting results concerning the use of variable shape parameters in the radial basis functions.

It is possible to apply the analysed reconstruction schemes directly to any location, possibly with less accuracy, since several methods are more accurate when the reconstruction points are located at the barycenters of the polygonal cells, getting worse near the edges, specially if only nearest neighbours are used in the reconstructions. With wider stencils, LSQ and RBF schemes do not loose accuracy in the reconstructions to arbitrary locations, although the reconstructed field will present discontinuities over the cell edges. An alternative, which ensures global continuity, is the use of the reconstruction algorithms to fixed points (e.g. barycenters or cell nodes), followed by a continuous interpolation from these values. With this latter approach, the order of the reconstruction scheme will be preserved if the interpolation method is chosen accordingly. We derived an extension to the sphere of the generalized Wachspress barycentric coordinates [15,16], sufficient to maintain second order accuracy and to ensure global continuity, which is adequate for the schemes considered in the paper.

Vector reconstructions may be used in different contexts, such as for data assimilation methods or visualization algorithms. Our principal interest is their use on semi-Lagrangian schemes, specially on tracer transport methods. Our focus is on traditional interpolating semi-Lagrangian schemes [10], although our analysis may be applicable to cell oriented conservative semi-Lagrangian transport schemes (as in [17] or [18]) which rely on accurate trajectory computations. We show that the requirements for a globally second order scheme involve the computation of trajectories with third order accuracy. A vector reconstruction method of at least second order will be necessary for that, if the trajectories are computed with traditional schemes such as Robert's method [19]. We tested a two-time-level semi-Lagrangian advection scheme on geodesic icosahedral C-grids. We analysed the precision of the trajectories approximations and the overall accuracy of the scheme, employing a deformational test flow suggested in [20]. The adequacy of the hybrid scheme for vector reconstructions has been demonstrated.

The paper is organized as follows. In Section 2 we analyse and extend existing vector reconstruction methods to spherical polygonal C-grids. Section 3 is devoted to the proposed hybrid reconstruction scheme, which is compared to the other methods, including tests on a locally refined Voronoi tessellation. Section 4 is dedicated to tests with the semi-Lagrangian advection method. We close the paper in Section 5 with some final remarks. The analysis of Perot's method on aligned cells is presented in Appendix A. Appendix B contains details about the addition of a polynomial part in RBF vector reconstruction methods. The accuracy requirements for our semi-Lagrangian scheme are derived in Appendix C.

2. Vector reconstruction methods

Let $\{u_i\}_{i=1,\dots,n}$ be the components of a vector field, $\vec{u}$, normal to given grid edges at their midpoints. The vector reconstruction problem is how to obtain a vector field $\vec{u}_h$ that approximates $\vec{u}$, based on the known values u_i. Usually $\vec{u}_h$ is required to satisfy the interpolatory conditions