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Inverse transport calculations in optical imaging with subspace optimization algorithms

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ABSTRACT

Inverse boundary value problems for the radiative transport equation play an important role in optics-based medical imaging techniques such as diffuse optical tomography (DOT) and fluorescence optical tomography (FOT). Despite the rapid progress in the mathematical theory and numerical computation of these inverse problems in recent years, developing robust and efficient reconstruction algorithms remains a challenging task and an active research topic. We propose here a robust reconstruction method that is based on subspace minimization techniques. The method splits the unknown transport solution (or a functional of it) into low-frequency and high-frequency components, and uses singular value decomposition to analytically recover part of low-frequency information. Minimization is then applied to recover part of the high-frequency components of the unknowns. We present some numerical simulations with synthetic data to demonstrate the performance of the proposed algorithm.

1. Introduction

The mathematical and computational study of inverse coefficient problems to the radiative transport equations has attracted extensive attention in recent years; see for instance the reviews [8,62–64,68] and references therein. The main objective of these inverse problems is to reconstruct physical parameters in the radiative transport equation from partial information on the solution to the equation. These inverse problems have important applications in many areas of science and engineering, such as ocean, atmospheric and interstellar optics [14,11,24,32,59,65,74,84], radiation therapy planning [3,16,38,43,60,73,80,82], diffuse optical tomography and quantitative photoacoustic tomography [1,3–5,12,20,29,35,39,40,50–52,55,61,70–72,76–79,85], molecular imaging [12,36,37,49,54,75] and many more [15,6,7,13,15,21–23,26,27,30,44–47,52,56,67,74,81,83,84,86].

We consider here the application of inverse transport problems in biomedical optical imaging techniques such as diffuse optical tomography (DOT) [1,3–5,12,20,29,35,39,40,52,70–72] and fluorescence optical tomography (FOT) [12,36,37,54] where radiative transport equations are often employed as the model for light propagation in biological tissues. To setup the problem, let us denote by $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ the tissue of interest, with sufficiently regular surface $\partial\Omega$. We denote by $\mathbb{S}^{d-1}$ the unit sphere in $\mathbb{R}^d$, and $\mathbf{v} \in \mathbb{S}^{d-1}$ the unit vector on the sphere. We denote by $X \equiv \Omega \times \mathbb{S}^{d-1}$ the phase space and define the boundary sets of the phase space, $\Gamma_\pm$, as

$$\Gamma_\pm = \left\{(\mathbf{x}, \mathbf{v}) \in \partial\Omega \times \mathbb{S}^{d-1} \text{ s.t. } \pm \mathbf{v} \cdot \mathbf{n}(\mathbf{x}) > 0\right\}$$

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with $\mathbf{n}(\mathbf{x})$ the unit outer normal vector at $\mathbf{x} \in \partial\Omega$. The radiative transport equation for the phase-space density distribution of the photons in the tissue can be written as

$$\begin{aligned} &\mathbf{v} \cdot \nabla u(\mathbf{x}, \mathbf{v}) + \sigma_{\mathfrak{a}}(\mathbf{x}) u(\mathbf{x}, \mathbf{v}) = \sigma_{\mathfrak{s}}(\mathbf{x}) \mathcal{K} u(\mathbf{x}, \mathbf{v}) \quad \text{in } X \\ &u(\mathbf{x}, \mathbf{v}) = f(\mathbf{x}) \quad \text{on } \Gamma_-, \end{aligned} \tag{1}$$

where $u(\mathbf{x}, \mathbf{v})$ is the density of photons at $\mathbf{x} \in \Omega$ traveling in direction $\mathbf{v}$, and f is the light source. The positive functions $\sigma_{\mathfrak{a}}(\mathbf{x})$ and $\sigma_{\mathfrak{s}}(\mathbf{x})$ are the absorption coefficient and the scattering coefficient respectively. The total absorption coefficient is given by $\sigma(\mathbf{x}) \equiv \sigma_{\mathfrak{a}}(\mathbf{x}) + \sigma_{\mathfrak{s}}(\mathbf{x})$. The scattering operator $\mathcal{K}$ is given by

$$\mathcal{K} u(\mathbf{x}, \mathbf{v}) = \int_{\mathbb{S}^{d-1}} k(\mathbf{v}, \mathbf{v}') u(\mathbf{x}, \mathbf{v}') d\mathbf{v}' - u(\mathbf{x}, \mathbf{v}) \tag{2}$$

where the scattering kernel $k(\mathbf{v}, \mathbf{v}')$ describes the probability that photons traveling in direction $\mathbf{v}'$ get scattered into direction $\mathbf{v}$. Note that to conserve the total mass, we have normalized the surface measure $d\mathbf{v}$ on $\mathbb{S}^{d-1}$ and the scattering kernel $k(\mathbf{v}, \mathbf{v}')$ such that

$$\int_{\mathbb{S}^{d-1}} d\mathbf{v} = 1, \quad \text{and} \quad \int_{\mathbb{S}^{d-1}} k(\mathbf{v}, \mathbf{v}') d\mathbf{v}' = 1, \ \forall \mathbf{v} \in \mathbb{S}^{d-1}. \tag{3}$$

In biomedical optics, the scattering kernel is often taken as the Henyey–Greenstein phase function [42]:

$$k(\mathbf{v}, \mathbf{v}') \equiv k_g(\mathbf{v} \cdot \mathbf{v}') = \Pi \frac{1 - g^2}{(1 + g^2 - 2g\mathbf{v} \cdot \mathbf{v}')^{d/2}}, \tag{4}$$

which is a one-parameter function that depends only on the angle between the two directions $\mathbf{v}$ and $\mathbf{v}'$ for a given anisotropy factor $g \in [-1, 1]$. The normalization constant Π is determined by the normalization condition (3).

The function $f(\mathbf{x})$ models the illumination source used in imaging experiments. In practical application of biomedical imaging, for instance in DOT and FOT, it is often technically difficult to construct angularly-resolved illumination sources. This is the main reason for us to employ an isotropic source function (referring to the fact that $f(\mathbf{x})$ does not depend on the angular variable $\mathbf{v}$) in the transport model (1). The measured data in biomedical optical imaging is usually a functional of the solution to the transport equation. Once again, due to the fact that it is difficult to measure angularly-resolved quantities, angularly-averaged quantities are usually measured. Here we consider applications (for instance DOT and FOT) where the measurement is the photon current on the surface of the tissue. The current is defined as

$$j(\mathbf{x}) \equiv \mathcal{M} u(\mathbf{x}) = \int_{\{\mathbf{v} \in \mathbb{S}^{d-1} : \mathbf{v} \cdot \mathbf{n}(\mathbf{x}) > 0\}} \mathbf{v} \cdot \mathbf{n}(\mathbf{x}) u(\mathbf{x}, \mathbf{v})_{|\Gamma_+} d\mathbf{v}, \quad \mathbf{x} \in \partial\Omega. \tag{5}$$

The objective of the biomedical imaging problems here is to reconstruct the optical absorption and scattering coefficients of biological tissues, $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{s}}$ from data encoded in the albedo operator:

$$\Lambda_{\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{s}}} : f(\mathbf{x}) \mapsto j(\mathbf{x}) \tag{6}$$

There are two major issues with diffuse optical imaging. The first issue is its low resolution due to the multiple scattering of light in tissues. Mathematically, this is manifested as the instability of the inverse transport problem [8,9]. By instability we mean that the noise in the data are significantly amplified in the inversion process, assuming that the problem admits a unique solution to start with. To stabilize the inverse problem, one can incorporate additional *a priori* information into the computational inversion algorithms. Commonly-used *a priori* information including, for instance, the smoothness or non-smoothness of the unknown [36,37,41] and the shape of the regions of interests [5,29]. The second issue with diffuse optical tomography is that there is no analytical inversion formulas for the image reconstruction problem, even in very academic geometrical configuration [72]. Computational reconstruction algorithms based on the radiative transport model are in general extremely slow. Fast reconstruction algorithms are actively sought by researchers in the field.

The instability of the inverse transport problems implies that when there is no available *a priori* information, only low-frequency components of the unknowns can be reconstructed stably. One should thus not spend too much efforts trying to reconstruct high-frequency components of the unknowns. Based on this observation and the idea of subspace minimization [25,66,87], we propose here a fast computational reconstruction method for the aforementioned inverse transport problems. Our method relies on the fact that we can explicitly factorize out some unstable components of the albedo operator $\Lambda_{\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{s}}}$ defined in (6). The unstable components of the albedo operator then impose a natural limit on the highest-frequency components of the unknown that can be reconstructed stably from the data. The factorization of $\Lambda_{\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{s}}}$ is not unique in general. For our purpose, we follow the ideas in [25,66,87] to reformulate the transport problem into the form

$$j = \mathcal{A}\mathfrak{u}, \tag{7}$$

$$\mathfrak{u} = \mathcal{B}\mathfrak{u} - \mathfrak{f}, \tag{8}$$