Journal of Computational Physics 273 (2014) 310–326

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

# A full discrete dispersion analysis of time-domain simulations of acoustic liners with flow

G. Gabard [a,*], E.J. Brambley [b]

[a] Institute of Sound and Vibration Research, University of Southampton, University Road, Southampton, SO17 1BJ, United Kingdom
[b] Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA, United Kingdom

ARTICLE INFO

*Article history:*
Received 25 November 2013
Received in revised form 24 March 2014
Accepted 2 May 2014
Available online 10 May 2014

*Keywords:*
Acoustic liner
Impedance
Myers
Time domain
Finite difference
Dispersion analysis

ABSTRACT

The effect of flow over an acoustic liner is generally described by the Myers impedance condition. The use of this impedance condition in time-domain numerical simulations has been plagued by stability issues, and various *ad hoc* techniques based on artificial damping or filtering have been used to stabilise the solution. The theoretical issue leading to the ill-posedness of this impedance condition in the time domain is now well understood. For computational models, some trends have been identified, but no detailed investigation of the cause of the instabilities in numerical simulations has been undertaken to date. This paper presents a dispersion analysis of the complete numerical model, based on finite-difference approximations, for a two-dimensional model of a uniform flow above an impedance surface. It provides insight into the properties of the instability in the numerical model and clarifies the parameters that influence its presence. The dispersion analysis is also used to give useful information about the accuracy of the acoustic solution.
Comparison between the dispersion analysis and numerical simulations shows that the instability associated with the Myers condition can be identified in the numerical model, but its properties differ significantly from that of the continuous model. The unbounded growth of the instability in the continuous model is not present in the numerical model due to the wavenumber aliasing inherent to numerical approximations. Instead, the numerical instability includes a wavenumber component behaving as an absolute instability. The trend previously reported that the instability is more likely to appear with fine grids is explained. While the instability in the numerical model is heavily dependent on the spatial resolution, it is well resolved in time and is not sensitive to the time step. In addition filtering techniques to stabilise the solutions are considered and it is found that, while they can reduce the instability in some cases, they do not represent a systematic or robust solution in general.



## 1. Introduction

Acoustic linings and the effects of their presence, position, and properties continue to be of major importance to sound reduction in aeroengines. The boundary condition often used to model acoustics in flow over acoustic linings is the Myers [27], or Ingard–Myers [20], boundary condition. This boundary condition, formulated in the frequency-domain, relates the acoustic pressure $\hat{p}e^{i\omega t}$ and acoustic normal velocity $\hat{v}_n e^{i\omega t}$ at the lining in order to give a specified lining impedance $\hat{Z}(\omega)$. The Myers boundary condition is given by

* Corresponding author.

http://dx.doi.org/10.1016/j.jcp.2014.05.004

$$\mathrm{i}\omega\hat{v}_n = \left[\mathrm{i}\omega + \mathbf{u}_0\cdot\nabla - (\mathbf{n}\cdot\nabla\mathbf{u}_0)\cdot\mathbf{n}\right]\frac{\hat{p}}{\hat{Z}}, \tag{1.1}$$

where $\mathbf{u}_0$ is the mean flow velocity and $\mathbf{n}$ is the lining surface normal. This boundary condition correctly represents the limit of a vanishingly-thin inviscid boundary layer over the acoustic lining [16,39], although the boundary layer needs to be extremely thin in some cases for this to be a good approximation [15]. A brief review of the mathematical modelling of acoustic linings with flow is given in Ref. [6].

Since the Myers boundary condition is naturally expressed in the frequency-domain, a number of techniques have been proposed for its adaption for use in time-domain numerical simulations for various different impedance models. Özyörük et al. [28] implemented the Myers boundary condition in a time-domain finite difference simulation, modelling the lining impedance $\hat{Z}(\omega)$ by a rational function of $\omega$ and applying a $z$-transform. Both Ju and Fung [21] and Li et al. [23] performed time-domain finite difference simulations modelling the acoustic lining using a modification of the three parameter model of Tam and Auriault [36] with the Myers boundary condition incorporated into an effective impedance; Li et al. [23] did so by modifying the impedance $\hat{Z}$, while Ju and Fung [21] modified the reflection coefficient $\hat{W} = (1-\hat{Z})/(1+\hat{Z})$. The three parameter acoustic lining model was subsequently implemented directly using the Myers boundary condition in a time-domain finite difference simulation by Tam et al. [37]. Chevaugeon et al. [12] implemented the Myers boundary condition in a time-domain discontinuous Galerkin simulation, modelling the acoustic lining as an Extended Helmholtz Resonator [33]. The Myers boundary condition with the Extended Helmholtz Resonator model may also be implemented in finite-difference simulations using a $z$-transform, as suggested by Rienstra [33] and demonstrated by Richter et al. [31].

A number of theoretical [32,10,5,34,24,7,25] and experimental [11,1,26] studies suggest that, in certain circumstances, flow over an acoustic lining should give rise to an instability. This physical instability could be one reason for the instability seen in time-domain simulations using the Myers boundary condition. However, Brambley [4] suggested the root cause of the numerical instability of the Myers boundary condition is that the underlying mathematical problem is ill-posed. This was demonstrated using a dispersion analysis, which considers waves of the form $\mathrm{e}^{\mathrm{i}\omega t-\mathrm{i}kx}$ where the frequency $\omega$ and wavenumber $k$ are linked by a dispersion relation $D(k,\omega)=0$. This dispersion relation may be solved to give $\omega$ as a function of $k$, which is equivalent to solving the initial value problem for the time evolution of initial conditions $\mathrm{e}^{-\mathrm{i}kx}$. For bounded initial conditions, implying $k$ is real, the stability of the subsequent evolution may be characterised by the exponential growth rate $-\mathrm{Im}(\omega(k))$. For the Myers boundary condition, Brambley [4] showed this growth rate to be unbounded as $k\to\infty$, implying that the problem is ill-posed. This is also the reason why a rigorous stability analysis is not possible for the Myers boundary condition, leading to the continuing debate over stability. However, to date, there has been no conclusive evidence directly linking the ill-posedness of the mathematical problem highlighted by Brambley [4] to the observed numerical instabilities in time-domain simulations.

In each of the computational studies mentioned above, filtering (or equivalently artificial dissipation) was found necessary to prevent high-frequency under-resolved oscillations and ensure stability. While it is known that filtering to remove high-frequency under-resolved oscillations is necessary with low-dispersion low-dissipation time-domain schemes [2], in practice the filtering needed to stabilise the Myers boundary condition is far greater than that needed to ensure stability within the bulk of the fluid; for example, Richter [29] found it necessary to use a very strong 3-point filter on part of the normal velocity $v_n$ prescribed by the Myers boundary condition. In general, the strength of filtering has to be adjusted on a case-by-case basis, since the strength of the instability varies with the grid resolution, the numerical scheme, the liner impedance $\hat{Z}$ and the flow velocity $\mathbf{u}_0$. In practice, therefore, it is difficult to ensure a stable simulation, and also difficult to assess the impact of the additional filtering on the prediction of sound attenuation by the lining. Moreover, use of this filtering implicitly assumes that flow over acoustic linings is stable, which may not reflect reality [11,1,26].

This paper investigates the numerical instability of time-domain simulations using the Myers boundary condition by performing a dispersion analysis of the discretised problem. The idea of a dispersion analysis of a discretization scheme is not new and a general description of this type of analyses can be found in [41]. Trefethen [40] gave a discussion of the dispersion relation for several finite-difference schemes applied to a 1D advection equation and a 2D wave equation, including the calculation and interpretation of the numerical group velocity. A similar concept was used by Tam and Webb [38] in creating Dispersion Relation Preserving (DRP) finite-difference schemes, where the finite-difference stencils were designed so that the numerical dispersion relation was a good approximation to the continuous dispersion relation of the continuous system being solved. More recently, Cossu and Loiseleux [13] considered the dispersion relation of three finite-difference schemes for solving the linearised Ginzburg–Landau equation, and interpreted their results in terms of convective and absolute instability by considering the numerical group velocity. However, each of these studies performed a dispersion analysis of only the spatial and temporal finite-difference schemes. In contrast, in this paper we perform a dispersion analysis of the entire numerical method, including not only the spatial and temporal finite-difference schemes but also the numerical implementation of the Myers boundary condition, the artificial filtering, and the non-reflecting boundary conditions at the edges of the computational domain. To our knowledge, this is the first time such a complete discretised dispersion analysis has been performed. This numerical dispersion analysis is used: (i) to compare the instability observed in time-domain simulations to the unstable mode predicted by the continuous model; (ii) to provide insight into the discrepancies between the finite-difference approximation and the continuous model; and (iii) to study the parameters of the numerical model that can influence the strength of the instability.

This paper is organised as follows. The benchmark problem and the numerical methods are described in the next section, and the presence of unstable solutions is illustrated. In Section 3 the dispersion analysis of the numerical model is