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An accurate, robust, and easy-to-implement method for integration over arbitrary polyhedra: Application to embedded interface methods

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ABSTRACT

We present an accurate method for the numerical integration of polynomials over arbitrary polyhedra. Using the divergence theorem, the method transforms the domain integral into integrals evaluated over the facets of the polyhedra. The necessity of performing symbolic computation during such transformation is eliminated by using one dimensional Gauss quadrature rule. The facet integrals are computed with the help of quadratures available for triangles and quadrilaterals. Numerical examples, in which the proposed method is used to integrate the weak form of the Navier–Stokes equations in an embedded interface method (EIM), are presented. The results show that our method is as accurate and generalized as the most widely used volume decomposition based methods. Moreover, since the method involves neither volume decomposition nor symbolic computations, it is much easier for computer implementation. Also, the present method is more efficient than other available integration methods based on the divergence theorem. Efficiency of the method is also compared with the volume decomposition based methods and moment fitting methods. To our knowledge, this is the first article that compares both accuracy and computational efficiency of methods relying on volume decomposition and those based on the divergence theorem.

1. Introduction

1.1. Objective and motivation

Consider the integration of a polynomial function $\mathcal{F}$ over the domain $\mathcal{R} \subset \mathbb{R}^n$

$$I_{\mathcal{R}} = \int_{\mathcal{R}} \mathcal{F} d\mathcal{R} \tag{1}$$

Evaluation of (1) is of great practical relevance when $\mathcal{R}$ is an arbitrary shaped polygon or polyhedron, since such integrations arise frequently in computational modeling and simulations in a variety of fields. Some notable applications

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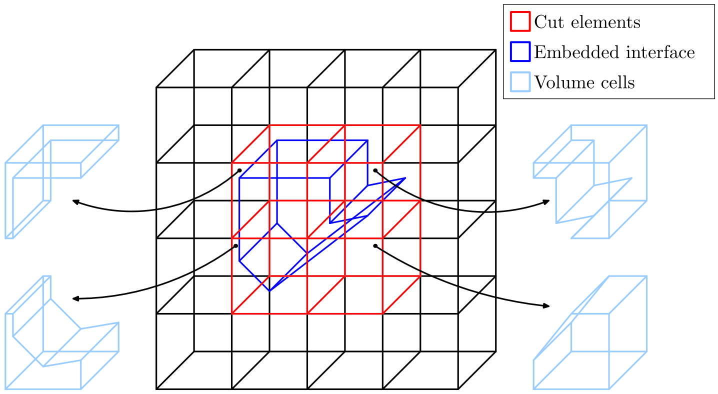

Fig. 1. Interface cut and the resulting complex polyhedral shaped volume cells for a small embedded interface method example.

include polygonal and polyhedral finite element methods [1–6], embedded interface methods [7–15], level set based methods [16–19], electronic structure calculations [20,21], aerospace applications [22–24], computational geometry [25–27] etc. The general characteristic of many of these applications is that the integrand $\mathcal{F}$ is not explicitly available, but is constructed during the solution process. As reviewed in the following text, most of the available methods to handle such integrations are either very complicated to implement or they are not accurate enough. Our objective is to devise an efficient and generalized, but easy-to-implement method to accurately approximate the above integral. The present work is motivated by our interest in developing an efficient embedded interface method (EIM) to handle complex fluid-structure interaction (FSI) and multiphase flow problems [13,18,28–31]. In EIM, evaluation of stiffness matrices over the elements that are cut by the interface requires computation of such integrals over arbitrary polyhedral shaped volume cells (weak form integration) as shown in Fig. 1. Accuracy of the weak form integration over these volume cells largely influences the overall solution accuracy. Moreover, in transient FSI and multiphase flow problems, since the interface changes its position and shape with time, the geometrical cutting operations and construction of integration method for the volume cells are performed at each time step. Hence, an accurate and efficient method to evaluate equation (1) over arbitrary polyhedra is an essential prerequisite to develop efficient EIMs.

1.2. Brief review of existing methods

Since the integration over polyhedra is important across a wide variety of fields, it is impossible to review all the available works. Hence we make note only of the methods that are related to the finite element literature.

Three important class of integration methods that are used in FEM are

1. Tessellation
2. Moment fitting methods
3. Methods based on the divergence theorem

In most works, the integration over a polyhedron is accomplished by the tessellation procedure, which involves a volume decomposition process [5,7,11,13,15,32]. First, the given polyhedron is decomposed into a number of tetrahedra. Then, the polynomial is integrated over each of these tetrahedra, and summed up to get the integral value over the polyhedron. Though tessellation yields accurate value of integrals, the associated volume decomposition procedure is complicated to implement, especially when one aims at robust methods for complex problems.

Other methods, which do not involve volume decomposition process, either solve the moment fitting equations or use the divergence theorem.

In moment fitting methods, to obtain a quadrature rule of order n, all the monomials $\phi = \{x^i y^j z^k, i + j + k \leq n\}$ are integrated over the polyhedra. Then, a quadrature rule is fit to integrate these monomials exactly [33–38]. The method used to integrate ϕ decides the applicability of such procedures. For example, in [36], Lasserre's method is used to integrate ϕ which limits the applicability of such a procedure to convex volumes. By using the divergence theorem to accomplish the integration of ϕ, the method is extended to deal with both convex and concave volumes in [37]. The drawbacks of this class of methods are the difficulty to obtain the location of quadrature points in three dimensional problems, and it is slightly less accurate when compared to tessellation when used for EIMs [37].

Various researchers have used the divergence theorem to perform integration over arbitrary polygons and polyhedra. The divergence theorem is used to simplify the domain integral into integration along the boundaries of the domain that can be easily evaluated. However, almost all the available methods of this kind either assume that the integrand of interest is predefined [6,25–27,39,40], or rely on symbolic computations to integrate generalized functions [1,41]. There are severe arguments against using these methods in a general finite element framework, since