Journal of Computational Physics 273 (2014) 572–588

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Discontinuous Galerkin methods for solving Boussinesq–Green–Naghdi equations in resolving non-linear and dispersive surface water waves

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ARTICLE INFO

Article history:
Received 15 July 2013
Received in revised form 18 April 2014
Accepted 23 May 2014
Available online 2 June 2014

Keywords:
Rotational Boussinesq–Green–Naghdi
Discontinuous Galerkin
Mixed space–time derivatives
Unstructured mesh
Surf-zone
Non-conservative

ABSTRACT

A local discontinuous Galerkin method for Boussinesq–Green–Naghdi equations is presented and validated against experimental results for wave transformation over a submerged shoal. Currently Green–Naghdi equations have many variants. In this paper a numerical method in one dimension is presented for the Green–Naghdi equations based on rotational characteristics in the velocity field. Stability criterion is also established for the linearized Green–Naghdi equations for both the analytical problem and the numerical method. Verification is done against a linearized standing wave problem in flat bathymetry and h, p (denoted by K in this paper) error rates are plotted. Validation plots show good agreement of the numerical results with the experimental ones.

1. Introduction

Surface water wave theory has been an evolving research topic where asymptotic models have been used to resolve wave characteristics. While the shallow water assumption is valid in regions where the characteristic wavelength exceeds a typical depth by orders of magnitude, Boussinesq-type equations have been used to model near-shore wave motion. The basic idea behind all the Boussinesq theories is the introduction of a polynomial representation of the velocity field in the vertical co-ordinate which reduces a $3D$ flow model to a $2D$ flow model. Most of these theories are based on an asymptotic extension of the additional shallow water physics into deeper water to arrive at inviscid, non-linear wave evolution equations. While many models [28,31,32], even though exhibiting good non-linear properties, have limited radius of convergence [24] and are restricted to a finite value of kh_b (k represents a typical wave-number; h_b represents a typical depth), recently some Boussinesq theories [27] have exhibited very high radius of convergence. However, since almost all Boussinesq based models assume an irrotational flow field they are valid up to the breaking point. Since vortices are generated from wave breaking, any model based on irrotational flow will induce large errors in the velocity field.

An alternate approach to the computation of shallow water nonlinear dispersive waves lies in the Green–Naghdi [19,34,36] formulation, where a polynomial structure for the velocity field is retained without any irrotational assumptions. Almost all Green–Naghdi based formulations have been developed in the shallow water limit, although researchers

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http://dx.doi.org/10.1016/j.jcp.2014.05.035
0021-9991/

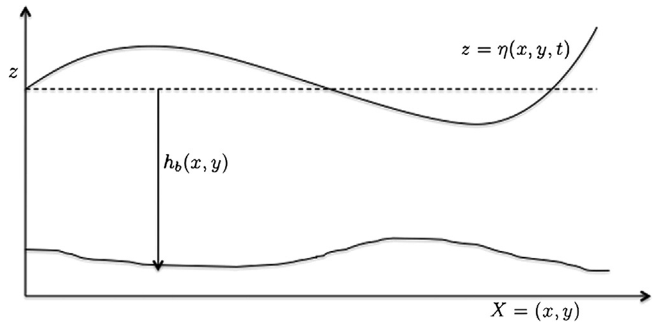

Fig. 1. Domain showing bathymetry and surface elevation.

[38] have successfully extended the formulation to deeper waters. Recently, in [40], the authors developed the Green–Naghdi formulation to arbitrary levels of approximation but also retained the Boussinesq scaling. Such a formulation can be naturally extended to model the surf-zone which is a highly energetic part of the near shore ocean where waves shoal, break and transmit energy to the shore [35,29,23]. Henceforth, in this paper, we refer to these equations as the **R-GN** equations. There are also water wave theories based on the Green–Naghdi approach that employ irrotational characteristics into the velocity formulation. Such systems have been known to provide accurate linear and non-linear dispersion [26,6], and their irrotational assumption brings it more in line with standard Boussinesq systems. We'll refer to these equations as the **I-GN** equations. Since our ultimate goal is to use Green–Naghdi equations to model surf-zone dynamics where we'll need rotational features in the velocity fields to include turbulent/viscous stresses; we'll mainly focus on the R-GN model. The numerical techniques developed for the **R-GN** equations will also apply to the **I-GN** equations.

While Green–Naghdi theory has been identified as a fairly accurate theory that captures non-linear dispersion, there aren't many numerical methods that can be used to solve such equations in an arbitrary grid. Part of this is due to the nature of the Green–Naghdi equations which contain non-linear products of higher-order spatial derivatives and mixed spatio-temporal derivatives. In this paper, we investigate a discontinuous Galerkin method [9] to solve both the R-GN and I-GN equations described above. Even though the R-GN equations are in non-conservative form and contain higher order derivatives, by the use of the local discontinuous Galerkin approximation we can easily handle non-linear products of the derivatives. Moreover, discontinuous Galerkin methods are known to handle complex geometry and extend easily to higher dimensions. Although recently a discontinuous Galerkin (DG) based method has been developed for high order Boussinesq equations [17], at present there are no DG or finite element methods for Green–Naghdi based equations.

In Section 2, we present the physical model, i.e. the R-GN equations due to [40]. Section 3 is devoted to the numerical discretization of these equations using the discontinuous Galerkin method for the spatial approximation. We outline the DG method and comment on the linear and non-linear stability of such an approximation. Finally, we carry out the verification and validation test of our numerical method and conclude with a summary and future work.

2. Governing equations: R-GN equations

Usually, in water wave theory one works with the *non-dimensional* Euler equations for an incompressible fluid. A typical domain is show in Fig. 1. The continuity equation reduces to the *free surface* equation given by,

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \int_{-h_b}^{\eta} \mathbf{u} dz = 0, \tag{2.1}$$

where $\eta = \eta(x, y, t)$ is the free surface. The non-dimensional momentum equations, in *Cartesian co-ordinates*, are given by

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + w \frac{\partial \mathbf{u}}{\partial z} + \nabla P = 0. \tag{2.2}$$

$$\mu^2 \frac{\partial w}{\partial t} + \mu^2 \mathbf{u} \cdot \nabla w + \mu^2 w \frac{\partial w}{\partial z} + \frac{\partial P}{\partial z} + g = 0, \tag{2.3}$$

where $\nabla = [\partial/\partial x, \partial/\partial y]^T$, $\mathbf{u} = [u, v]^T$ and μ represents a dimensionless wave number. Integrating (2.3) from z to η, and assuming a zero gauge pressure at the free surface, we find

$$P(z) = \mu^2 \int_z^{\eta} \frac{\partial w}{\partial t} dz + \mu^2 \int_z^{\eta} \mathbf{u} \cdot \nabla w dz + \mu^2 \int_z^{\eta} w \frac{\partial w}{\partial z} dz + g(\eta - z). \tag{2.4}$$

In accordance with the classical Boussinesq and Green–Naghdi theory, we follow the recipe outlined in [40] where an approximated velocity field given by