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ADER schemes and high order coupling on networks of hyperbolic conservation laws

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ABSTRACT

In this article we present a method to extend high order finite volume schemes to networks of hyperbolic conservation laws with algebraic coupling conditions. This method is based on an ADER approach in time to solve the generalized Riemann problem at the junction. Additionally to the high order accuracy, this approach maintains an exact conservation of quantities if stated by the coupling conditions. Several numerical examples confirm the benefits of a high order coupling procedure for high order accuracy and stable shock capturing.

1. Introduction

Networks of hyperbolic conservation laws occur in many applications such as the human circulatory system [1–4], gas pipelines [5–7], water [8–11] and road networks [12,13]. For all these applications accurate and stable numerical methods are needed.

In the past decades high order accurate numerical methods for hyperbolic conservation laws have been developed, such as WENO- [14–17] or ADER-schemes [18–26]. These methods have proven their efficiency in many challenging applications [27–29].

For networks of hyperbolic conservation laws the flow across the edges can be dealt with by any appropriate numerical method for standard conservation laws [22]. Special attention has to be given to the coupling conditions. The direct solving of the coupling conditions only provides first order information, which can either be used directly by applying a Godunov scheme [6,30] or to fill corresponding ghost cells [9,31] at the boundary. A second order approach is studied in [32].

In this article we present an approach to incorporate the coupling conditions numerically up to an arbitrary order of accuracy. This includes the computation of the flux across the outer boundary as well as the reconstruction of ghost cell values for numerical methods of higher order. Therefore we apply an ADER approach in time to the algebraic coupling conditions. This can be used to solve the generalized Riemann problems at the junctions providing time dependent data at the junction. These can be reformulated as spatial data by the inverse Cauchy–Kowalewski procedure to fill the ghost cells for the numerical method along the edges.

This paper is organized as follows. First, the first order Godunov solver at the junction is recalled. Second, the generalized Riemann Problem at the junction is discussed. These two ingredients can be used for high order spatial reconstruction at

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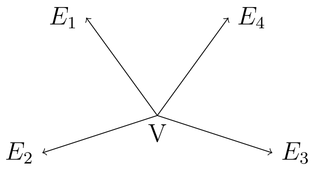

Fig. 1. Edge orientation convention.

the nodes, which leads to a high order numerical method for the complete network. For this approach we prove that the quantities conserved by the algebraic coupling conditions are also conserved by the numerical method. Further we show that for a simple 1 to 1 coupling the presented method coincides with a classical ADER scheme on a single continuous line. In the numerical examples we study the order accuracy for test cases with smooth data and show the need of a high order coupling procedure. Finally we investigate the stability in case of shock waves and the applicability for large networks of conservation laws.

2. High order coupling procedure

2.1. Notations

A network $\mathcal{N} = (\mathcal{E}, \mathcal{V})$ consists of a set of edges $\mathcal{E}$ and a set of connecting vertices $\mathcal{V}$,

$$\mathcal{E} = \{E_1, \dots, E_{\tilde{n}}\}, \qquad \mathcal{V} = \{V_1, \dots, V_{\tilde{m}}\}.$$

On each edge E_i, $i = 1, \dots, \tilde{n}$, we consider the quantities $u^i(x,t) \in \mathbb{R}^{d_i}$, which are governed by a hyperbolic conservation law

$$\partial_t u^i + \partial_x f^i\big(u^i\big) = 0, \tag{1}$$

with the flux function $f^i : \mathbb{R}^{d_i} \to \mathbb{R}^{d_i}$, the time $t \in \mathbb{R}^+$ and location $x \in [0, L_i]$. At every vertex V_j, the functions u^i are coupled via c^j algebraic coupling conditions given by $\Phi^j : \bigotimes_{i=1}^n \mathbb{R}^{d_i} \to \mathbb{R}^{c^j}$ for n connected edges. In order to ease the notation in the following we consider only a single junction without index and assume that all $n = \tilde{n}$ connected edges are oriented outwards, see Fig. 1 for a schematic. Thus the coupling point in each edge is located at $x = 0$ allowing us to drop the spatial variable in the context of the coupling conditions

$$\Phi\big(u^1(t), \dots, u^n(t)\big) = 0, \quad u^i(t) = u^i(0,t). \tag{2}$$

Following the results of [33,9], the number of coupling conditions c has to coincide with the number of characteristics running out of the vertex. In order to maintain a fixed number of coupling conditions over time, we require for each edge i the eigenvalues λ^i_j, $j = 1, \dots, d_i$, of the Jacobian ∇f^i to be bounded away from zero by some constant $\tilde{\epsilon} > 0$

$$\lambda^i_1 \le \dots \le \lambda^i_{d_i}, \qquad \big|\lambda^i_j\big| > \tilde{\epsilon} \quad \forall j = 1, \dots, d_i. \tag{3}$$

Finally the following condition guarantees that to each outgoing characteristic exactly one value can be assigned and thus the well-posedness of the coupling conditions is given by

$$\det\big(D_{u^1}\Phi\big(u^1_g, \dots, u^n_g\big)R^1 | \dots | D_{u^n}\Phi\big(u^1_g, \dots, u^n_g\big)R^n\big) \ne 0, \tag{4}$$

where $R^i = [r^i_{d_i - c_i + 1} | \dots | r^i_{d_i}]$ is the collection of all eigenvectors associated with positive eigenvalues of ∇f^i. c_i is the number of positive eigenvalues in the edge i and $\sum_{i=1}^n c_i = c$ holds.

2.1.1. Examples

Throughout this paper we will consider the isentropic Euler equations as example

$$\begin{aligned} &\partial_t \rho + \partial_x q = 0 \\ &\partial_t q + \partial_x\left(\frac{q^2}{\rho} + p(\rho)\right) = 0, \end{aligned} \tag{5}$$

with the density ρ, density flux q and a pressure law $p : \mathbb{R}^+ \to \mathbb{R}^+$. Commonly used coupling conditions in this context for subsonic flow, $\frac{q}{\rho} < \sqrt{\partial_\rho p(\rho)}$, are the following two variants.

Definition 1. Pressure coupling [6,31]:

$$\begin{aligned} &\sum_{i=1}^n q_i = 0 \\ &p_1(\rho_1) - p_i(\rho_i) = 0 \quad 2 \le i \le n. \end{aligned} \tag{6}$$

In case of an identical pressure law in all connected edges, the last $n-1$ equations reduce to $\rho_1 - \rho_i = 0$, $2 \le i \le n$.