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Optimization-based limiters for the spectral element method

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ABSTRACT

We introduce a new family of optimization based limiters for the h–p spectral element method. The native spectral element advection operator is oscillatory, but due to its mimetic properties it is locally conservative and has a monotone property with respect to element averages. We exploit this property to construct locally conservative quasimonotone and sign-preserving limiters. The quasimonotone limiter prevents all overshoots and undershoots at the element level, but is not strictly non-oscillatory. It also maintains quasimonotonicity even with the addition of a dissipation term such as viscosity or hyperviscosity. The limiters are based on a least-squares formulation with equality and inequality constraints and are local to each element. We evaluate the new limiters using a deformational flow test case for advection on the surface of the sphere. We focus on mesh refinement for moderate ($p = 3$) and high order ($p = 6$) elements. As expected, the spectral element method obtains its formal order of accuracy for smooth problems without limiters. For advection of fields with cusps and discontinuities, the high order convergence is lost, but in all cases, $p = 6$ outperforms $p = 3$ for the same degrees of freedom.

1. Introduction

The spectral element method (SEM) is a continuous Galerkin finite element method which uses a Gauss–Lobatto quadrature based inner product [1]. It is h–p capable, relies on globally continuous polynomial basis functions and the equations of interest are solved in integral form. The unique feature of the method is that the quadrature based inner product allows the construction of compactly supported, globally continuous basis and test functions that are orthogonal, leading to a diagonal mass matrix. Because of this, the SEM can use explicit time-integrators without the added expense of having to solve a global matrix inverse problem. This makes the method very efficient for the type of PDEs encountered in time-dependent geophysical problems. It has proven accurate and effective for global atmospheric circulation modeling [2–10], ocean modeling [11,12], and planetary-scale seismology [13]. It also has unsurpassed parallel performance, including Gordon Bell awards [14–16] and scales to hundreds of thousands of processors in the Community Atmosphere Model [17,18].

One caveat of the SEM finds its source in the advection operator. For advection, the SEM can achieve excellent accuracy in the L_2-norm since it uses relatively high-degree polynomials (in our case, degrees 3 and 6). However, the fact that the basis functions are globally continuous makes it difficult to preserve discrete analogs of other important properties of advection such as monotonicity and positivity. Traditional SEM results are quite oscillatory [19]. In this work we show how

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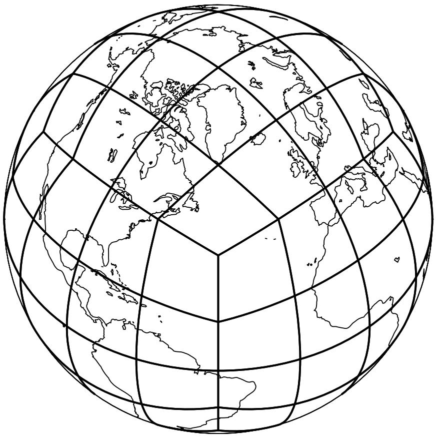

Fig. 1. Tiling the surface of the sphere with quadrilaterals. An inscribed cube is projected to the surface of the sphere. The faces of the cubed-sphere are further subdivided to form a quadrilateral grid of the desired resolution. The gnomonic equal angle projection is used, resulting in a quasi-uniform but non-orthogonal grid [44].

to incorporate a local element reconstruction for the mimetic SEM formulation which yields an efficient quasimonotone limiter. We define quasimonotone to be monotone with respect to the spectral element nodal values. The limiter is bounds preserving for the advection equation with or without the addition of diffusion or hyperviscosity.

Continuous-Galerkin finite element methods have only recently been discovered to be locally conservative [20]. This fact, combined with the finite element method's long history with unstructured meshes has made them competitive with finite volume (FV) and discontinuous-Galerkin (DG) methods for solving transport problems in complex geometries. There has been extensive work on limiters for FV and DG methods, many examples for FV are surveyed in [21] and for DG in [22]. There has been much less work on limiters for continuous-Galerkin finite-element methods. Recent work includes [23,24]. In [23], a generalization of the flux corrected transport approach is taken: first, one constructs a highly dissipative but truly monotone advection step by ensuring the linear advection operator has all positive entries (following [25]). This operator is used to compute the monotonicity constraints that are then applied to the results of the unlimited advection operator.

Here we describe a different approach that takes advantage of two key features of the SEM. The first feature is the SEM's discrete divergence theorem that comes from the mimetic nature of the method [26]. Unlike FV and DG methods, the SEM does not have direct control over the element flux, but the discrete divergence theorem shows that there is an implied element flux, and this flux naturally satisfies a monotonicity property with respect to element averages. The second feature is the diagonal mass matrix which leads to the monotonicity-preserving direct stiffness summation (DSS) procedure. In our formulation, the DSS procedure is written as a projection operator. Thus the limiters can be applied locally within each element, and their monotonicity properties will be preserved during the application of the DSS. The limiters presented here involve solving a constrained optimization problem that is completely local to each element. The only additional interelement communication introduced is in determining the suitable minimum and maximum constraints. Thus these limiters achieve the similar high computational/communication ratio as the SEM and preserve the SEM's scalability on parallel computers.

The focus of this work is to use these features of the SEM to develop a local bounds-preserving limiter which preserves the SEM's excellent scalability. In light of the results in [27], such schemes cannot also preserve the SEM's high-order accuracy. We confirm these results with convergence studies, but we also show that the accuracy is competitive with that obtained by other approaches.

Below we first give a condensed summary of the SEM in curvilinear geometry in order to establish our notation. We follow this with a proof of the element-average monotonicity property of the SEM, subject to a CFL-like restriction on the timestep. We then describe the sign-preserving and quasimonotone limiter and associated optimization problem. Finally, we present numerical results using a recently proposed suite of advection test cases for spherical geometry [28,29]. This standardized test case has been used to evaluate many proposed numerical methods for atmospheric tracer transport [30].

2. The spectral element method

2.1. Functional spaces

Let Ω represent our computational domain. We first mesh Ω using a quadrilateral finite-element mesh with M elements denoted $\{\Omega_m\}_{m=1}^{M}$. Here the focus is on the case where Ω is the surface of the sphere, and we employ a cubed-sphere based tiling of the sphere with quadrilaterals as shown in Fig. 1. It is assumed that the mesh has no hanging nodes, and that each element is C^1 mapped to the reference element $[-1,1]\times[-1,1]$. We denote this map and its inverse by $\vec{r}=\vec{r}(\vec{x};m)$ and