Journal of Computational Physics 267 (2014) 196–209

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Manufactured solutions for the three-dimensional Euler equations with relevance to Inertial Confinement Fusion ☆

J. Waltz [a,*], T.R. Canfield [b], N.R. Morgan [a], L.D. Risinger [c], J.G. Wohlbier [c]

[a] Computational Physics Division, Los Alamos National Laboratory, Los Alamos, NM, United States
[b] Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM, United States
[c] Computational and Computer Sciences Division, Los Alamos National Laboratory, Los Alamos, NM, United States

ARTICLE INFO

Article history:
Received 14 February 2013
Received in revised form 18 February 2014
Accepted 21 February 2014
Available online 6 March 2014

Keywords:
ICF modeling
Hydrodynamics
Method of manufactured solutions

ABSTRACT

We present a set of manufactured solutions for the three-dimensional (3D) Euler equations. The purpose of these solutions is to allow for code verification against true 3D flows with physical relevance, as opposed to 3D simulations of lower-dimensional problems or manufactured solutions that lack physical relevance. Of particular interest are solutions with relevance to Inertial Confinement Fusion (ICF) capsules. While ICF capsules are designed for spherical symmetry, they are hypothesized to become highly 3D at late time due to phenomena such as Rayleigh–Taylor instability, drive asymmetry, and vortex decay. ICF capsules also involve highly nonlinear coupling between the fluid dynamics and other physics, such as radiation transport and thermonuclear fusion. The manufactured solutions we present are specifically designed to test the terms and couplings in the Euler equations that are relevant to these phenomena. Example numerical results generated with a 3D Finite Element hydrodynamics code are presented, including mesh convergence studies.

1. Introduction

Code verification is a fundamental part of establishing confidence in complex simulation codes.[1] Formal code verification typically involves convergence studies using analytical solutions to the equations of interest. For the equations of fluid dynamics, *i.e.* the Navier–Stokes or Euler equations, these test problems are typically defined for one-dimensional (1D) and in some cases two-dimensional (2D) flows. Common examples are the Riemann (shock tube) problem (see *e.g.* [2]), the Sedov problem [3], the 2D Taylor–Green vortex (see *e.g.* [4]), and circular pipe flow [5].

Formal verification of three-dimensional (3D) simulation codes is often performed with 3D calculations of 1D or 2D problems. This practice adds value in that it tests a code's ability to calculate basic solution structure as well as lower-dimensional phenomena that may be embedded in a higher-dimensional space. The extent to which a 3D computed solution deviates from a 1D or 2D analytic result also can highlight errors in implementation and deficiencies in discrete operators. However, this practice also is inherently limited because it does not test a code's ability to calculate a true 3D solution. Although some analytical solutions to the 3D Euler and Navier–Stokes equations do exist (*e.g.* [6–8]), typical codes are

☆ This work was supported by the U.S. Department of Energy through the LANL LDRD Program (grant No. DE-AC52-06NA25396). LANL Report no. LA-UR-13-20185.

* Corresponding author.
E-mail address: jwaltz@lanl.gov (J. Waltz).

[1] We adopt the terminology of Roache [1] and take "verification" to mean the process of determining that the underlying equations are solved correctly.

http://dx.doi.org/10.1016/j.jcp.2014.02.040
0021-9991/

not designed to support the often complicated mathematical forms of these solutions. These solutions also may not have physical relevance to the particular applications of interest.

The Method of Manufactured Solutions (MMS) [9] is an alternative approach to code verification. In the MMS, a solution form is constructed *a priori*, and corresponding source terms are obtained by substitution of the solution into the governing equations. The MMS has been used by many authors in fluid dynamics (*e.g.* [10–13]) as well as fluid-structure interaction [14], electromagnetism [15], radiation transport [16], and radiation-hydrodynamics [17]. However, the majority of these works present only 1D and 2D solutions, even when the simulation code under consideration is actually 3D. The use of MMS in this fashion is therefore fundamentally no different than the use of 1D and 2D analytical solutions.

The purpose of this work is to present a set of 3D manufactured solutions for the compressible Euler equations that complement typical verification studies and thereby enhance the formal verification basis of 3D codes. To maximize applicability, the solutions are formulated strictly in terms of primitive fluid variables and an ideal gas equation of state. This allows use with typical 3D simulation codes, *i.e.* those based on canonical Finite Difference (FD), Finite Volume (FV), or Finite Element (FE) methods with approximate Riemann solvers, explicit time integration, and standard Dirichlet and Neumann boundary conditions.

Although solutions generated with the MMS need not have physical relevance, the solutions presented herein are designed to explore specific aspects of the governing equations and thereby more closely replicate the fluid dynamics present in the intended applications. Mathematically, this design goal implies that the dominant terms and truncation errors in the test problem should be similar to those in the application. While this procedure cannot replace code validation – *i.e.* comparison against experimental data in the intended regime – it increases the likelihood that verification testing will reveal deficiencies in a code's ability to model relevant phenomena. For example, if a particular application involves a highly nonlinear time dependence, a manufactured solution with a similar time dependence is more likely to expose stability and accuracy issues than a manufactured solution with a weak time dependence. The significance of this form of targeted verification with respect to real problems is that it increases confidence of the numerical algorithms in the physical conditions of interest – as opposed to verifying the numerical algorithms in one set of conditions and applying them in another.

In the remainder of this paper we summarize the equations of interest, briefly describe the test code, and present three manufactured solutions. Numerical results for each problem along with spatial convergence studies also are included.

2. Governing equations

The governing equations under consideration are the 3D unsteady Euler equations, written here in the flux-conservative form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = S_\rho \tag{1}$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j + p\delta_{ij}) = S_{u,i} \tag{2}$$

$$\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x_i}\big[u_i(\rho E + p)\big] = S_E \tag{3}$$

where in the usual notation ρ is the density, u_i the velocity vector, $E = u_i u_i/2 + e$ the specific total energy, e the specific internal energy, and p the pressure. S_ρ, $S_{u,i}$, and S_E are the source terms that arise from the application of the MMS. The system is closed with the ideal gas law equation of state

$$p = \rho e(\gamma - 1) \tag{4}$$

where γ is the ratio of specific heats. Although the exact form of the equation of state is of secondary importance for this work, the use of the ideal gas law simplifies the derivations that follow; in principle any analytic equation of state could be used.

Of particular interest for this work are three phenomena that arise in the above system, namely:

- Vortical flow, which is hypothesized to drive the development of turbulence in ICF capsules [18];
- Nonlinear energy growth, which arises in both radiation-hydrodynamics and thermonuclear fusion [19]; and
- Rayleigh–Taylor unstable conditions, which also are believed to play a role in the development of mixing in ICF capsules [20,21]

A manufactured solution for each of these phenomena was constructed from appropriate analytical forms for a subset of the six independent variables in the system (1)–(4). Various mathematical constraints were then applied to determine solutions for the remaining variables. The derived source terms resulted from back substitution of the defined solutions and corresponding derivatives into (1)–(4). Note that because the solutions are smooth by construction, they do not contain shocks or contact discontinuities. On the other hand, smooth solutions allow code verification at full theoretical accuracy without the reduction to first-order that occurs at discontinuities for monotone schemes.