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Fokas integral equations for three dimensional layered-media scattering

David M. Ambrose [a], David P. Nicholls [b,*]

[a] Department of Mathematics, Drexel University, Philadelphia, PA 19104, United States
[b] Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL 60607, United States

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ABSTRACT

The scattering of acoustic waves by periodic structures is of central importance in a wide range of problems of scientific and technological interest. This paper describes a rapid, high-order numerical algorithm for simulating solutions of Helmholtz equations coupled across irregular (non-trivial) interfaces meant to model acoustic waves incident upon a multiply layered medium. Building upon an interfacial formulation from previous work, we describe an Integral Equation strategy inspired by recent developments of Fokas and collaborators for its numerical approximation. The method requires only the discretization of the layer interfaces (so that the number of unknowns is an order of magnitude smaller than volumetric approaches), while it requires neither specialized quadrature rules nor periodized fundamental solutions characteristic of many popular Boundary Integral/Element Methods. As with previous contributions by the authors on this formulation, this approach is efficient and spectrally accurate for smooth interfaces.

1. Introduction

The interaction of acoustic waves with periodic structures plays an important role in many scientific problems. From remote sensing [31] to underwater acoustics [2], the ability to robustly simulate scattered fields with high accuracy is of fundamental importance. Here we focus upon the high-order numerical simulation of solutions of Helmholtz equations coupled across irregular (non-trivial) interfaces meant to model acoustic waves in a multiply layered medium. Based upon a surface formulation recently developed by the author [19], we present a novel Integral Equation Method inspired by recent developments of Fokas and collaborators [1,9,29,30].

Many volumetric numerical algorithms have been devised for the simulation of these problems, for instance, Finite Differences (see, e.g., [26]), Finite Elements (see, e.g., [33]), and Spectral Elements (see, e.g., [13]). These methods suffer from the requirement that they discretize the full volume of the problem domain which results in both a prohibitive number of degrees of freedom, and also the difficult question of appropriately specifying a far-field boundary condition explicitly.

Surface methods are an appealing alternative and those based upon Boundary Integrals (BIM) or Boundary Elements (BEM) are very popular (see, e.g., [28]). In fact, the approach we advocate here falls precisely into this category. These BIM/BEM require only discretization of the layer *interfaces* (rather than the whole structure) and, due to the choice of the Green's function, satisfy the far-field boundary condition *exactly*. While these methods can deliver high-accuracy simulations with greatly reduced operation counts, there are several difficulties which need to be addressed [27]. First, high-order simulations can only be realized with specially designed quadrature rules which respect the singularities in the Green's

* Corresponding author.

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function (and its derivative, in certain formulations). Additionally, BIM/BEM typically give rise to dense linear systems to be solved which require carefully designed preconditioned iterative methods (with accelerated matrix–vector products, e.g., by the Fast-Multipole Method [11]) for configurations of engineering interest. Finally, for periodic structures the Green's function must be periodized which greatly increases the computational cost.

Before addressing these concerns as they impact our own formulation, we note that Boundary Perturbation Methods (BPM) emerged as an appealing strategy which maintain the reduced numbers of degrees of freedom of BIM/BEM while avoiding the need for special quadrature formulas or preconditioned iterative solution procedures for dense systems. Among these are: (i) the Method of Field Expansions due to Bruno and Reitich [3–5] for doubly layered media, and the generalization of Malcolm and Nicholls [15,19] to multiply-layered structures; and (ii) the Method of Operator Expansions due to Milder [17,18] (see also improvements in [6]) which was generalized to multiple layers by Malcolm and Nicholls [14,19].

Returning to the challenges faced by BIM/BEM mentioned above, in this contribution we utilize Fokas' approach to discovering Integral Equations (which we term Fokas Integral Equations – FIE) satisfied by the Dirichlet–Neumann Operator (DNO) and its corresponding Dirichlet data. These formulas do *not* involve the fundamental solution, but rather smooth, "conjugated" solutions of the quasi-periodic Helmholtz problem meaning that simple quadrature rules (e.g., Nyström's Method) may be utilized while periodization is unnecessary. In addition, due to use of a clever alternative to the standard Green's Identity, the *derivative* of the interface shapes never appear in our FIEs meaning that configurations of rather low smoothness can be accommodated in comparison with alternative approaches (see Appendix A for one choice). The density of the linear systems to be solved cannot be avoided, however, this is somewhat ameliorated by the fact that the number of degrees of freedom required is often quite modest due to the high-order accuracy of our quadratures, and as derivatives of the layer shapes never appear in our integral relations. Finally, the conditioning properties of these FIEs has recently been called into question (see, e.g., the preprint of Wilkening and Vasan [32]) and, as we discuss in Remark 5.2, it can challenge the effectiveness of such methods. However, for problems of small to moderate size, we have found that the remarkable simplicity and speed of the current algorithm cannot be matched by alternative strategies.

Turning to layered media scattering, we pair these new FIE relationships to the interfacial formulation of such problems recently devised by one of the authors [19]. The resulting algorithm has the speed and efficiency of a boundary method without the complications of iterative linear solvers, Green's function periodization algorithms, or the derivation and implementation of perturbation recursions. One simply builds a linear system of equations with readily computed values and solves with any standard algorithm (e.g., Gaussian elimination).

The rest of the paper is organized as follows: In Section 2 we recall the governing equations of layered media scattering, and a surface formulation in Section 2.1 (with special cases discussed in Section 2.2). In Section 3 we introduce our new (Fokas) Integral Equations with relations for the top layer in Section 3.1, the bottom layer in Section 3.2, and middle layers in Section 3.3 (we summarize these formulas and the zero-perturbation case in Section 3.4). We discuss formulas for computing the efficiencies in Section 4, and numerical results in Section 5. We present a class of exact (non-plane-wave) solutions in Section 5.1 and numerical implementation and error measurement details in Section 5.2. We close with convergence studies in Section 5.3 and plane-wave simulations in Section 5.4.

2. Governing equations

Consider a $d = (d_1, d_2)$-periodic, multiply-layered material with M interfaces at

$$y = \bar{g}^{(m)} + g^{(m)}(x_1, x_2) = \bar{g}^{(m)} + g^{(m)}(x), \quad 1 \le m \le M,$$

where $x = (x_1, x_2)$, $\bar{g}^{(m)}$ are constants, and

$$g^{(m)}(x + d) = g^{(m)}(x_1 + d_1, x_2 + d_2) = g^{(m)}(x_1, x_2) = g^{(m)}(x).$$

These interfaces separate $(M + 1)$-many layers which define the domains

$$\begin{aligned} S^{(0)} &:= \{ y > \bar{g}^{(1)} + g^{(1)}(x) \} \\ S^{(m)} &:= \{ \bar{g}^{(m+1)} + g^{(m+1)}(x) < y < \bar{g}^{(m)} + g^{(m)}(x) \} \quad 1 \le m \le M - 1 \\ S^{(M)} &:= \{ y < \bar{g}^{(M)} + g^{(M)}(x) \}, \end{aligned}$$

with (upward pointing) normals $N^{(m)} := (-\nabla_x g^{(m)}, 1)^T$; see Fig. 1. In each layer we assume a constant speed $c^{(m)}$ and that the structure is insonified from above by plane-wave incidence

$$u^i(x, y, t) = e^{-i\omega t} e^{i(\alpha \cdot x - \beta y)} = e^{-i\omega t} v^i(x, y), \quad \alpha = (\alpha_1, \alpha_2)^T.$$

In each layer the quantity $k^{(m)} = \omega / c^{(m)}$ specifies the properties of the material and the frequency of radiation common to the incident and scattered field in the structure. It is well-known [25] that the problem can be restated as a time-harmonic one of time-independent *reduced* scattered fields, $v^{(m)}(x, y)$, which, in each layer, are α-quasiperiodic

$$v^{(m)}(x + d, y) = e^{i(\alpha \cdot d)} v^{(m)}(x, y),$$