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Nonlinear Fourier analysis for discontinuous conductivities: Computational results

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ABSTRACT

Two reconstruction methods of Electrical Impedance Tomography (EIT) are numerically compared for nonsmooth conductivities in the plane based on the use of complex geometrical optics (CGO) solutions to D-bar equations involving the global uniqueness proofs for Calderón problem exposed in Nachman (1996) [43] and Astala and Päivärinta (2006) [6]: the Astala–Päivärinta theory-based *low-pass transport matrix method* implemented in Astala et al. (2011) [3] and the *shortcut method* which considers ingredients of both theories. The latter method is formally similar to the Nachman theory-based regularized EIT reconstruction algorithm studied in Knudsen et al. (2009) [34] and several references from there.

New numerical results are presented using parallel computation with size parameters larger than ever, leading mainly to two conclusions as follows. First, both methods can approximate piecewise constant conductivities better and better as the cutoff frequency increases, and there seems to be a Gibbs-like phenomenon producing ringing artifacts. Second, the transport matrix method loses accuracy away from a (freely chosen) pivot point located outside of the object to be studied, whereas the shortcut method produces reconstructions with more uniform quality.

1. Introduction

We study a widely applicable nonlinear Fourier transform in dimension two. We perform numerical tests related to the nonlinear Gibbs phenomenon with much larger cutoff frequencies than before. Furthermore, we compare two computational inverse transformations, called *low-pass transport matrix method* and *shortcut method* in terms of accuracy.

The inverse conductivity problem of Calderón [10] is the main source of applications of the nonlinear Fourier transform we consider. Let $\Omega \subset \mathbb{R}^2$ be the unit disc and let $\sigma : \Omega \to (0, \infty)$ be an essentially bounded measurable function satisfying $\sigma(x) \geq c > 0$ for almost every $x \in \Omega$. Let $u \in H^1(\Omega)$ be the unique solution to the following elliptic Dirichlet problem:

$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$u|_{\partial\Omega} = \phi \in H^{1/2}(\partial\Omega). \tag{1.2}$$

The inverse conductivity problem consists on recovering σ from the Dirichlet-to-Neumann (DN) map or voltage-to-current map defined by

$$\Lambda_\sigma : \phi \mapsto \sigma \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega}.$$

Here ν is the unit outer normal to the boundary. Note that the map $\Lambda : \sigma \mapsto \Lambda_\sigma$ is nonlinear.

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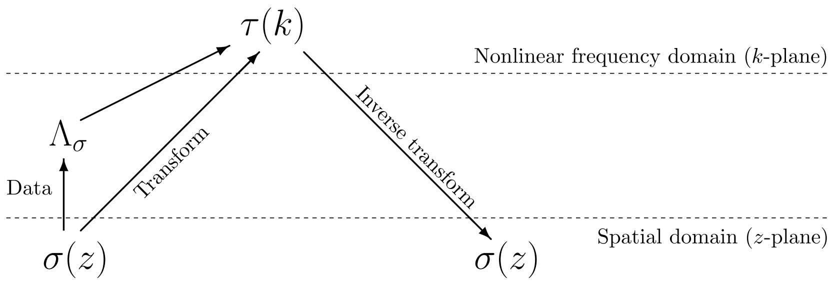
The inverse conductivity problem is related to many practical applications, including the medical imaging technique called *electrical impedance tomography* (EIT). There one attaches electrodes to the skin of a patient, feeds electric currents into the body and measures the resulting voltages at the electrodes. Repeating the measurement with several current patterns yields a current-to-voltage data matrix that can be used to compute an approximation Λ_σ^δ to Λ_σ. Since different organs and tissues have different conductivities, recovering σ computationally from Λ_σ^δ amounts to creating an image of the inner structure of the patient. See [12,41] for more information on EIT and its applications.

Recovering σ from Λ_σ^δ is a nonlinear and ill-posed inverse problem, whose computational solution requires regularization. Several categories of solution methods have been suggested and tested in the literature; in this work we focus on so-called D-bar methods based on complex geometrical optics (CGO) solutions. There are three main flavors of D-bar methods for EIT:

- Schrödinger equation approach for twice differentiable σ. Introduced by Nachman in 1996 [43], implemented numerically in [30,31,34,40,41,45].
- First-order system approach for once differentiable σ. Introduced by Brown and Uhlmann in 1997 [7], implemented numerically in [19,32,36].
- Beltrami equation approach assuming no smoothness ($\sigma \in L^\infty(\Omega)$). Introduced by Astala and Päivärinta in 2006 [6], implemented numerically in [3–5,41]. The assumption $\sigma \in L^\infty(\Omega)$ was the one originally used by Calderón in [10].

Using these approaches, a number of conditional stability results have been studied for the Calderón problem. The most recent results in the plane were obtained by Clop, Faraco and Ruiz in [13], where stability in L^2-norm was proven for conductivities on Lipschitz domains in the fractional Sobolev spaces $W^{\alpha,p}$ with $\alpha > 0$, $1 < p < \infty$, and in [15], where the Lipschitz condition on the boundary of the domain was removed. In dimension $d \geq 3$, conditional stability in Hölder norm for just $C^{1+\varepsilon}$ conductivities on bounded Lipschitz domains was proved by Caro, García and the third author in [11] using the method presented in [18].

The three aforementioned D-bar methods for EIT in the two-dimensional case are based on the use of nonlinear Fourier transforms specially adapted to the inverse conductivity problem. Schematically, the idea looks like this:



The main point above is that the nonlinear Fourier transform can be calculated from the infinite-precision data Λ_σ, typically via solving a second-kind Fredholm boundary integral equation for the traces of the CGO solutions on $\partial\Omega$.

In practice one is not given the infinite-precision data Λ_σ, but rather the noisy and finite-dimensional approximation Λ_σ^δ. Typically all we know about Λ_σ^δ is that $\|\Lambda_\sigma - \Lambda_\sigma^\delta\|_Y < \delta$ for some (known) noise level $\delta > 0$ measured in an appropriate norm $\|\cdot\|_Y$. Most CGO-based EIT methods need to be regularized by a truncation $|k| < R$ in the nonlinear frequency-domain as illustrated in Fig. 1.

The regularization step results in a smooth reconstruction. This smoothing property of the nonlinear low-pass filter resembles qualitatively the effect of linear low-pass filtering of images. In particular, the smaller R is, the blurrier the reconstruction becomes.

The cut-off frequency R is determined by the noise amplitude δ, and typically R cannot exceed 7 in practical situations. However, it is interesting to understand how the conductivity is represented by its nonlinear Fourier transform. To study this numerically, we compute nonlinear Fourier transforms in large discs such as $|k| < 60$ and compute low-pass filtered reconstructions using inverse nonlinear Fourier transform and observe the results.

We compare two reconstruction methods based on the use of the CGO solutions (2.1): the *low-pass transport matrix method* implemented in [3], and a *shortcut method* based on solving a D-bar equation. The latter method does not have rigorous analysis available yet, but it is formally similar to the regularized EIT reconstruction algorithm studied in [30,31,33,34,40,43,45].

Our new computational findings can be roughly summarized by the following two points. First, both methods can approximate piecewise constant conductivities better and better as the cutoff frequency R increases, and there seems to be a Gibbs-like phenomenon producing ringing artifacts. Second, the transport matrix method loses accuracy away from a (freely chosen) pivot point located outside of $\overline{\Omega}$, whereas the shortcut method produces reconstructions with more uniform quality.

Fig. 2 shows some of our numerical results via the *shortcut method* for a nonsymmetric conductivity distribution.

The rotationally symmetric examples presented in Section 4.1 provide numerical evidence of the fact that discontinuous conductivities can be reconstructed with the shortcut method more and more accurately when R tends to infinity.