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Up to sixth-order accurate A-stable implicit schemes applied to the Discontinuous Galerkin discretized Navier–Stokes equations

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A B S T R A C T

In this paper a high-order implicit multi-step method, known in the literature as Two Implicit Advanced Step-point (TIAS) method, has been implemented in a high-order Discontinuous Galerkin (DG) solver for the unsteady Euler and Navier–Stokes equations.
Application of the absolute stability condition to this class of multi-step multi-stage time discretization methods allowed to determine formulae coefficients which ensure A-stability up to order 6. The stability properties of such schemes have been verified by considering linear model problems. The dispersion and dissipation errors introduced by TIAS method have been investigated by looking at the analytical solution of the oscillation equation.
The performance of the high-order accurate, both in space and time, TIAS-DG scheme has been evaluated by computing three test cases: an isentropic convecting vortex under two different testing conditions and a laminar vortex shedding behind a circular cylinder. To illustrate the effectiveness and the advantages of the proposed high-order time discretization, the results of the fourth- and sixth-order accurate TIAS schemes have been compared with the results obtained using the standard second-order accurate Backward Differentiation Formula, BDF2, and the five stage fourth-order accurate Strong Stability Preserving Runge–Kutta scheme, SSPRK4.

1. Introduction

Unsteady flow computations are becoming commonplace in many applications of Computational Fluid Dynamics (CFD). In this context the potential of recent high-order methods, such as the Discontinuous Galerkin (DG) method, appears very promising, [1–3]. Explicit high-order Runge–Kutta schemes combined with the DG space discretization (RK-DG methods), [2,4,5], were initially used to address the numerical solution of unsteady flow problems. Explicit methods are well suited for problems with similar spatial and temporal scales, but become inefficient for unsteady flows of low reduced frequency, as well as for steady-state and/or stiff problems. In fact, the time step restriction of explicit methods applied to high-order spatial discretizations becomes more restrictive as the spatial accuracy increases. Due to their better stability properties,

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implicit methods are expected to provide a more effective way of solving problems with disparate temporal and spatial scales.

Many authors have considered implicit time integration schemes for the DG numerical solution of unsteady flows. A first example of a second-order Runge–Kutta scheme applied to the unsteady RANS equations was presented in [6]. This and other higher-order methods belonging to the class of linearly implicit, Rosenbrock-type, Runge–Kutta methods, have been subsequently investigated in [7,8] and in [9,10]. Considering several implicit time integration methods, Wang and Mavriplis [11] compared the performance of DIRK, BDF and trapezoidal time integration schemes applied to the Euler equations. Dolejsi [12] and Persson and Peraire [13] investigated BDF time integration schemes applied to the Euler and Navier–Stokes equations, respectively. Kanevsky et al. [14] suggested to use implicit–explicit (IMEX) schemes both for inviscid and viscous flows. While all these papers refer to two dimensional problems, Birken et al. [15,16] considered ESDIRK and SDIRK methods for the three-dimensional Navier–Stokes equations and compared their performance with that of explicit Runge–Kutta methods and of the Space–Time-Expansion DG of Lorcher et al. [17].

In this work we focus on the Two Implicit Advanced Step-point (TIAS) method presented by Cash and Psihoyios [18,19]. This implicit formula belongs to the wider family of Implicit Advanced Step-point methods. Extended and Modified Extended Backward Differentiation Formulae (MEBDF) proposed by Cash [20,21] are two important classes of these schemes. Following a similar approach this new multi-step and multi-stage method has been proposed based on the use of a second advanced step-point and thus an additional step forward. Therefore, TIAS methods have three predictors and an extra function evaluation per step, but better stability properties than EBDF and MEBDF methods. The use of two super-future points guarantees an A-stable formula up to order 6 and A(α)-stable up to order 9 [22,23]. TIAS schemes involve four stages: the first three are predictor stages that use a standard k-step BDF scheme, the last one is a corrector stage that uses an advanced implicit k-step formula of order $k+1$. The theoretical aspects of this approach were investigated in detail in [22]. We point out that although four nonlinear stages have to be solved per time step, the multistage predictor–corrector structure of the method allows i) to obtain accurate starting solutions of the Newton iterations ii) to efficiently implement the schemes in a variable step/variable order algorithm, making TIAS of interest for complex applications where residual and jacobian evaluations usually dominate the total computational cost. As concerns the stepsize and order adaption, it seems natural to implement these formulae using the fixed-coefficient approach, where the time step size and the changes in order are selected accordingly with the estimation of the time integration error. For the k-step TIAS method, this can be done comparing the local truncation error (LTE) of the embedded predictor BDF schemes, of order k, with the LTE of the corrector TIAS formula, of order $k+1$. For more details on both the way in which the local error of the lower order scheme can be approximated and the strategies adopted for efficient time-step and order changing refer to [24,25].

The first part of our paper is concerned with the numerical construction of high-order accurate A-stable TIAS schemes and the assessment of their stability, diffusion and dispersion properties by considering linear model problems. Furthermore, the relevant implementation issues of the proposed time integration method are discussed in detail. In the second part, we present numerical results obtained using this advanced multi-step method applied to a high-order DG discretization of the two-dimensional compressible Navier–Stokes equations [26].

The performance of the TIAS-DG scheme are evaluated by means of three test cases: an inviscid isentropic convecting vortex under two different testing conditions and a laminar vortex shedding behind a circular cylinder. To illustrate the advantages of the high-order time discretization, the results of the fourth- and sixth-order accurate TIAS scheme will be compared with the results obtained with the standard second-order accurate Backward Differentiation Formula, BDF2 [27], and the five stage fourth-order accurate Strong Stability Preserving Runge–Kutta scheme, SSPRK4 [28], using the same spatial discretization.

In the following of the paper the governing equations are presented in Section 2. Section 3 briefly discusses the space (DG) discretization. In Section 4, we describe the time (TIAS) discretization and give some relevant implementation details. An analysis of the TIAS method is presented in Section 5. Numerical results are showed and discussed in Section 6. Conclusions are finally reported in Section 7.

2. Governing equations

The two dimensional, compressible Navier–Stokes equations in conservative form based on the set of conservative variables $\mathbf{u} = [\rho, \rho v_1, \rho v_2, \rho E]^T$ are:

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{F}^c(\mathbf{u}) - \nabla \cdot \mathbf{F}^v(\mathbf{u}, \nabla \mathbf{u}) = 0, \tag{1}$$

where $\mathbf{F}^c = (\mathbf{f}_1^c, \mathbf{f}_2^c)^T$ and $\mathbf{F}^v = (\mathbf{f}_1^v, \mathbf{f}_2^v)^T$ are the inviscid and viscous flux vectors respectively, given by:

$$\mathbf{f}_j^c = \begin{pmatrix} \rho v_j \\ \rho v_1 v_j + p\delta_{1j} \\ \rho v_2 v_j + p\delta_{2j} \\ \rho H v_j \end{pmatrix}, \qquad \mathbf{f}_j^v = \begin{pmatrix} 0 \\ \tau_{1j} \\ \tau_{2j} \\ \tau_{ji} v_i - q_j \end{pmatrix}, \quad j = 1, 2.$$