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A discontinuous Galerkin discretization for solving the two-dimensional gas dynamics equations written under total Lagrangian formulation on general unstructured grids

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ABSTRACT

Based on the total Lagrangian kinematical description, a discontinuous Galerkin (DG) discretization of the gas dynamics equations is developed for two-dimensional fluid flows on general unstructured grids. Contrary to the updated Lagrangian formulation, which refers to the current moving configuration of the flow, the total Lagrangian formulation refers to the fixed reference configuration, which is usually the initial one. In this framework, the Lagrangian and Eulerian descriptions of the kinematical and the physical variables are related by means of the Piola transformation. Here, we describe a cell-centered high-order DG discretization of the physical conservation laws. The geometrical conservation law, which governs the time evolution of the deformation gradient, is solved by means of a finite element discretization. This approach allows to satisfy exactly the Piola compatibility condition. Regarding the DG approach, it relies on the use of a polynomial space approximation which is spanned by a Taylor basis. The main advantage in using this type of basis relies on its adaptability regardless the shape of the cell. The numerical fluxes at the cell interfaces are computed employing a node-based solver which can be viewed as an approximate Riemann solver. We present numerical results to illustrate the robustness and the accuracy up to third-order of our DG method. First, we show its ability to accurately capture geometrical features of a flow region employing curvilinear grids. Second, we demonstrate the dramatic improvement in symmetry preservation for radial flows.

1. Introduction

We aim at describing a high-order discontinuous Galerkin (DG) method for solving the two-dimensional total Lagrangian form of the gas dynamics equations on general unstructured grids. It is well known that fluid dynamics relies on two kinematics descriptions: the Eulerian or spatial description and the Lagrangian or material description, refer for instance to [25,22]. In the former, the conservation laws are written using a fixed reference frame whereas in the latter they are written through the use of a time dependent reference frame that follows the fluid motion. The Lagrangian representation is particularly well adapted to describe the time evolution of fluid flows contained in regions undergoing large shape changes

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due to strong compressions or expansions. Further, in this approach, there is no mass flux across the boundary surface of a control volume moving with the fluid velocity. Thus, Lagrangian representation provides a natural framework to track accurately material interfaces in multi-material compressible flows. Moreover, such a representation avoids the apparition of numerical diffusion resulting from the discretization of the convection terms present in the Eulerian frame.

This paper is primarily concerned with the development of a Lagrangian method for which the main feature relies on the use of the total Lagrangian formalism. In this approach, the physical conservation laws are written employing the Lagrangian coordinates which refer to the initial configuration of the fluid flow. Moreover, in these equations the divergence and gradient operators are expressed by means of the Piola transformation [25], which requires the knowledge of the deformation gradient tensor, i.e. the Jacobian matrix associated to the Lagrange–Euler flow map. The deformation gradient tensor characterizes the time evolving deformation and is governed by a partial differential equation named the geometric conservation law (GCL). To ensure the consistency between the initial and the current configurations, the deformation gradient tensor has to satisfy an involutive constraint [45], which implies the Piola compatibility condition. The total Lagrangian approach is very well known in the solid mechanics community wherein it is extensively used to model solid dynamics undergoing large deformations [25]. The first application of the total Lagrangian approach to the gas dynamics equations has been undertaken in [1,34] by means of a DG type discretization. However, the use of the aforementioned method is restricted to a representation on the initial configuration since it cannot be rigorously re-interpreted on the current configuration. We also note that the theoretical properties of the gas dynamics equations written under the total Lagrangian formulation have been thoroughly studied in [16,42].

In contrast with respect to the total Lagrangian formulation, the updated Lagrangian formulation is a moving domain method, which is widely employed. In this approach, the gas dynamics equations are written employing the Eulerian coordinates. They refer to the current configuration of the fluid flow. The time derivative of the physical variables is taken following the fluid particles paths: this is the material derivative. The integral formulation of the conservation laws is readily obtained by employing the Reynolds transport formula over an arbitrary moving control volume. The time rate of change of a zone volume is governed by a partial differential equation which is the updated Lagrangian form of the GCL. It is worth mentioning that at the discrete level the zone volume computed from its vertices coordinates must rigorously coincide with the zone volume deduced from the numerical solution of the GCL. This critical requirement is the cornerstone on which any proper multi-dimensional updated Lagrangian scheme should rely.

Two approaches are mainly employed to solve the updated Lagrangian formulation of the gas dynamics equations. The first one, which is called the staggered grid hydrodynamics, consists in using a staggered discretization wherein the kinematic variables (vertex position, velocity) are located at nodes whereas the thermodynamic variables (density, pressure, internal energy) are defined at the cell centers. The conversion of kinetic energy into internal energy through shock waves, consistently with the second law of thermodynamics, is ensured by adding an artificial viscosity term. The staggered grid schemes employed in most hydro-codes have been remarkably successful over the past decades in solving complex multi-dimensional compressible fluid flows, refer for instance to [9,10]. However, they clearly have some theoretical and practical deficiencies such as mesh imprinting and symmetry breaking. In addition, the fact that all variables are not conserved over the same space can lead to serious difficulties in the perspective of an arbitrary Lagrangian–Eulerian (ALE) extension. The second approach, known as cell-centered hydrodynamics, employs a cell-centered placement of all hydrodynamic variables including the momentum. This approach consists of a moving mesh finite volume method wherein the numerical fluxes (multi-valued nodal pressures and nodal velocity) are computed through the use of node-centered approximate Riemann solvers. In this framework, momentum and total energy are conserved and an entropy inequality is satisfied at the semi-discrete level to ensure the thermodynamic consistency of the numerical method. Moreover, the numerical fluxes are constructed to satisfy the GCL compatibility. The interested readers may refer to the following papers [11,37,12,38,3,8] for a more detailed description of this approach and its variants. Let us point out that work has been done to investigate the relationships between the staggered and the cell-centered discretizations, refer to [41,35].

Up to our knowledge, the interpretation of the staggered schemes of Goad [23] and Wilkins [53] by means of a finite element method has been initially introduced by Lascaux at the beginning of the 70s [32,33]. This finite element approach has been further developed, producing various interesting staggered schemes. For instance, a compatible finite element Lagrangian hydrodynamics algorithm used in a multi-material ALE strategy has been described in [2]. We also note the development of a variational multi-scale stabilized approach in finite element computation of Lagrangian hydrodynamics where a piecewise linear approximation was adopted for the variables [47,46]. The case of Q1/P0 finite element is studied in [48] where the kinematic variables are represented using a piecewise linear continuous approximation while the thermodynamic variables utilize a piecewise constant representation.

Except the pioneering work of [1,34], all the aforementioned approaches are characterized by an accuracy which is at most of second order, for problems with higher than one dimension space. This accuracy restriction is a natural consequence of the spatial discretization of the Lagrange–Euler flow map employed. Namely, the gas dynamics equations are discretized on a moving grid made of polygonal cells whose edges remain straight lines throughout the motion. This amounts to claim that the Lagrange–Euler flow map admits a linear continuous representation with respect to Eulerian coordinates over the deforming computational grid. Further, the kinematic velocity field also admits a linear continuous representation. Therefore, as noticed in [13], this approximation of the grid motion implies a second-order error in the numerical method. To reach a higher order of accuracy, one has to take into account a higher order discretization of the kinematics of the flow. This point has been successfully addressed in [13] in which the authors present a third-order Lagrangian scheme for solving gas