Journal of Computational Physics 276 (2014) 235–251

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Some numerical algorithms for solving the highly oscillatory second-order initial value problems ☆

Wenjie Liu, Boying Wu, Jiebao Sun *

Department of Mathematics, Harbin Institute of Technology, Harbin 150001, PR China

ARTICLE INFO

Article history:
Received 9 January 2014
Received in revised form 16 May 2014
Accepted 15 July 2014
Available online 28 July 2014

Keywords:
Highly oscillatory problems
Second-order initial value problems
Block spectral collocation method
Block boundary value method
Implicit Runge–Kutta method
Diagonally implicit Runge–Kutta method
Total variation diminishing Runge–Kutta method

ABSTRACT

In this paper, some numerical algorithms (spectral collocation method, block spectral collocation method, boundary value method, block boundary value method, implicit Runge–Kutta method, diagonally implicit Runge–Kutta method and total variation diminishing Runge–Kutta method) are used to solve the highly oscillatory second-order initial value problems. We first derive these methods for the first-order initial value problems, and then extend these methods to the highly oscillatory nonlinear systems by matrix analysis methods. These new methods preserve the accuracy of the original methods and the main advantages of these new methods are low storage requirements and high efficiency. Extensive numerical results are presented to demonstrate the convergence properties of these methods.

1. Introduction

In this paper, we study the spectral collocation method (or the block spectral collocation method) [1–5], the boundary value method (or the block boundary value method) [6–8], the implicit Runge–Kutta method [9–11], the diagonally implicit Runge–Kutta method [9,11,12] and the total variation diminishing (TVD) Runge–Kutta method [13,14] for the following highly oscillatory second-order initial value problem

$$\begin{cases} y''(t) = f\big(t, y(t), y'(t)\big), & t_0 < t \le T, \\ y(t_0) = y_0, & y'(t_0) = y'_0. \end{cases} \tag{1}$$

The problem (1) is the fundamental model in physics for describing mechanical vibrations behavior [15]. This model also appears in circuit simulations, flexible body dynamics and various quantum dynamics calculations [16–18]. Therefore, the construction of efficient numerical schemes for solving (1) is an important task.

Recently, several numerical schemes have been developed for (1), and some useful approaches to construct Runge–Kutta–Nyström (RKN) type methods have also been proposed. For details we refer to the monograph [16].

The block spectral collocation method (or the spectral collocation method), the block boundary value method (or the boundary value method), the implicit Runge–Kutta method, the diagonally implicit Runge–Kutta method and the TVD

☆ This work is partially supported by the National Science Foundation of China (11271100, 11301113) and the Fundamental Research Funds for the Central Universities (Grant No. HIT. IBRSEM. A. 201412), Harbin Science and Technology Innovative Talents Project of Special Fund (2013RFXYJ044).

* Corresponding author.
E-mail address: sunjiebao@hit.edu.cn (J. Sun).

http://dx.doi.org/10.1016/j.jcp.2014.07.033
0021-9991/

Runge–Kutta method are high-accuracy schemes for the first-order nonlinear initial value problems. So, we extend these methods for solving the second-order initial value problems by matrix analysis methods [19]. The main advantages of these new methods are low storage requirements and high efficiency.

The outline of this paper is as follows: in Section 2, we describe the spectral collocation method (SCM) and the block spectral collocation method (BSCM). The boundary value method (BVM) and the block boundary value method (BBVM) are briefly discussed in Section 3. In Section 4, we present the implicit Runge–Kutta (IRK) method and the diagonally implicit Runge–Kutta (DIRK) method. The TVD Runge–Kutta (TVDRK) method is described in Section 5. In Section 6, we discuss the stability of our methods. In Section 7, we provide some extensive numerical results to assess the convergence and accuracy of our methods. Finally, we summarize the main features of our methods and briefly comment on the extension in Section 8.

2. The spectral collocation method

In this section, we briefly introduce the spectral collocation method for solving the nonlinear system of second-order ODEs.

We introduce the Chebyshev–Gauss–Lobatto points in $\Lambda = [-1, 1]$,

$$\tilde{x}_j = \cos\left(\frac{j\pi}{N}\right), \quad j = 0, 1, \cdots, N.$$

By differentiating the polynomial and evaluating the polynomial at the same collocation points with $\tilde{f}_k = \tilde{f}(\tilde{x}_k)$, we have

$$\tilde{F}_N(\tilde{x}) = \sum_{j=0}^{N} \tilde{f}_k \tilde{L}_k(\tilde{x}), \tag{2}$$

where $\tilde{L}_k(\tilde{x})$ are the Lagrange basis polynomials given by (see [5])

$$\tilde{L}_k(\tilde{x}) = \frac{(-1)^{k+1}(1-\tilde{x}^2)T_N'(\tilde{x})}{c_k N^2(\tilde{x}-\tilde{x}_k)}, \quad k = 0, 1, \cdots, N, \tag{3}$$

where $T_N(\tilde{x}) = \cos(N\cos^{-1}\tilde{x})$ is the Chebyshev polynomial and

$$c_k = \begin{cases} 2, & k = 0 \text{ or } N, \\ 1, & \text{otherwise.} \end{cases}$$

Let $\tilde{F} = [\tilde{f}(\tilde{x}_0), \cdots, \tilde{f}(\tilde{x}_N)]^T$, $\tilde{F}^{(m)} = [\tilde{f}^{(m)}(\tilde{x}_0), \cdots, \tilde{f}^{(m)}(\tilde{x}_N)]^T$ and approximate the derivative of $\tilde{F}$ at $\tilde{x}_j$ by differentiating and evaluating (2), we get

$$\tilde{F}_N^{(m)}(\tilde{x}) = \sum_{j=0}^{N} \tilde{f}_k \tilde{L}_k^{(m)}(\tilde{x}), \quad m = 1, 2, \cdots. \tag{4}$$

Then (4) is equivalent to the following matrix equation

$$\tilde{F}^{(m)} = \tilde{D}^{(m)}\tilde{F}, \quad m = 1, 2, \cdots,$$

where $\tilde{D}^{(m)}$ is the $(m+1)\times(m+1)$ matrix whose entries are given by

$$\tilde{D}_{jk}^{(m)} = \tilde{L}_k^{(m)}(\tilde{x}_j), \quad j, k = 0, 1, \cdots, N.$$

The first-order Chebyshev differentiation matrix $\tilde{D}^{(1)} = \tilde{D} = (\tilde{d}_{kj})$ is given by (see [1–5])

$$\tilde{d}_{kj} = \begin{cases} -\frac{c_k}{2c_j}\frac{(-1)^{j+k}}{\sin((k+j)\frac{\pi}{2N})\sin((k-j)\frac{\pi}{2N})}, & k \neq j, \\ -\frac{1}{2}\cos(\frac{k\pi}{N})(1+\cot^2(\frac{k\pi}{N})), & k = j, k \neq 0, N, \\ \tilde{d}_{00} = -\tilde{d}_{NN} = \frac{2N^2+1}{6}. \end{cases} \tag{5}$$

Higher derivative matrices can be obtained as matrix powers, i.e.,

$$\tilde{D}^{(m)} = \left(\tilde{D}^{(1)}\right)^m.$$

Let $x_j = a - \frac{b-a}{2}(\tilde{x}_j - 1)$ be the Chebyshev–Gauss–Lobatto points in $[a, b]$, such that

$$\tilde{x}_j = 1 - \frac{2}{b-a}(x_j - a),$$

$$F^{(m)} = D^{(m)}F, \quad m = 1, 2, \cdots,$$