Journal of Computational Physics 276 (2014) 252–290

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Isotropic finite volume discretization

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ARTICLE INFO

Article history:
Received 7 December 2013
Received in revised form 28 June 2014
Accepted 16 July 2014
Available online 24 July 2014

Keywords:
Finite volume method
Isotropic discretization
Rotational invariance

ABSTRACT

Finite volume methods traditionally employ dimension by dimension extension of the one-dimensional reconstruction and averaging procedures to achieve spatial discretization of the governing partial differential equations on a structured Cartesian mesh in multiple dimensions. This simple approach based on tensor product stencils introduces an undesirable grid orientation dependence in the computed solution. The resulting anisotropic errors lead to a disparity in the calculations that is most prominent between directions parallel and diagonal to the grid lines. In this work we develop isotropic finite volume discretization schemes which minimize such grid orientation effects in multidimensional calculations by eliminating the directional bias in the lowest order term in the truncation error. Explicit isotropic expressions that relate the cell face averaged line and surface integrals of a function and its derivatives to the given cell area and volume averages are derived in two and three dimensions, respectively. It is found that a family of isotropic approximations with a free parameter can be derived by combining isotropic schemes based on next-nearest and next-next-nearest neighbors in three dimensions. Use of these isotropic expressions alone in a standard finite volume framework, however, is found to be insufficient in enforcing rotational invariance when the flux vector is nonlinear and/or spatially non-uniform. The rotationally invariant terms which lead to a loss of isotropy in such cases are explicitly identified and recast in a differential form. Various forms of flux correction terms which allow for a full recovery of rotational invariance in the lowest order truncation error terms, while preserving the formal order of accuracy and discrete conservation of the original finite volume method, are developed. Numerical tests in two and three dimensions attest the superior directional attributes of the proposed isotropic finite volume method. Prominent anisotropic errors, such as spurious asymmetric distortions on a circular reaction–diffusion wave that feature in the conventional finite volume implementation are effectively suppressed through isotropic finite volume discretization. Furthermore, for a given spatial resolution, a striking improvement in the prediction of kinetic energy decay rate corresponding to a general two-dimensional incompressible flow field is observed with the use of an isotropic finite volume method instead of the conventional discretization.

1. Introduction

Numerical methods that extend one-dimensional approximations to higher dimensions by employing tensor product polynomials and stencils offer a straightforward and reliable route towards performing multidimensional spatial discretization on a structured Cartesian mesh. Due to their simplicity such methods have gained popularity and find widespread

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http://dx.doi.org/10.1016/j.jcp.2014.07.025

use in multidimensional finite difference and finite volume Cartesian grid formulations [1,2]. The localized stencil associated with such multidimensional methods facilitates computationally efficient calculations in that the total cost associated with such methods increases only sublinearly with the number of dimensions. The dimensionally decoupled formulation employed in such methods also simplifies implementation of high-order discretization schemes since the basis polynomials used in the approximation are easily obtained by combining one-dimensional basis polynomials along the Cartesian axes.

While the reliability and effectiveness of the tensor product based combination of the one-dimensional approximation methods for multidimensional discretization is beyond doubt there are certain disadvantages of this simplified approach. Application of tensor product stencils and associated basis functions is known to introduce directional dependence in multidimensional calculations. The anisotropic errors that are caused by this directional dependence are often manifest in the computed solution in the form of spurious features with an imprint of the underlying Cartesian mesh. Formation and growth of anisotropic distortions on a radially symmetric expanding or contracting wave front is a well known example (e.g. [3–5], also see Section 5) which demonstrates how anisotropic errors induce an artificial asymmetry in the numerical solutions which should evolve symmetrically. The inability of the dimensionally decoupled conventional approach in preserving the directional attributes can severely impact the accuracy of the computed solution especially when the spatial resolution is such that the solution is either marginally resolved or under resolved. In such cases, the large growth in anisotropic errors that results from an inadequate spatial resolution can lead to a pronounced discrepancy between the directional attributes of the true physical solution and the one that is obtained from simulations.

To an extent this limitation can be overcome by increasing either the spatial resolution or the order of accuracy of the discretization methods (e.g. [5,6]). However, an improvement in the directional accuracy due either to the increased resolution or the order of accuracy comes at the price of a substantial rise in computational expense owing to the significant increase in the number of degrees of freedom and more stringent temporal stability restrictions for explicit time integration. Furthermore, a large class of interesting problems involve sharp gradients which can not be represented with sufficient accuracy on a discrete mesh even with successive refinements in spatial resolution or an increase in the order of accuracy. Discontinuity capturing simulations of formation and propagation of sharp fronts such as shock waves or formation of high curvature regions associated with merger or fragmentation events in multiphase flows constitute two such representative situations in which even an initially smooth solution loses its regularity at later times. In such cases any finite resolution provided by a fixed Eulerian mesh eventually becomes insufficient for a complete resolution of all the length scales associated with the physical solution. Even smooth solutions with high wavenumber content, such as the steep gradients in turbulent flows at sufficiently high Reynolds numbers, are essentially seen as near singular on a discrete mesh. In such situations, an improvement in directional accuracy with grid refinement is either impossible or prohibitively expensive. Moreover, the susceptibility of high-order discretization methods to numerical instabilities necessitates regularization in the form of filtering or artificial dissipation. This additional step is counterproductive in that it makes the task of controlling anisotropic errors much more difficult and can also lead to a substantial increase in computational expense. It is therefore necessary to devise alternative discretization strategies which eliminate anisotropic errors in the conventional dimensionally decoupled methods on relatively coarse meshes without resorting to either grid refinement or an increase in the formal order of accuracy.

Our prime objective in this work is to develop genuinely multidimensional approximation techniques which minimize anisotropic errors for a class of popular cell centered conservative finite volume methods. Finite volume methods have a distinct advantage over the conventional finite difference methods in that they naturally satisfy the primary conservation properties of the governing equations exactly. This attractive attribute is indispensable for multiphase flow computations where emphasis is placed on mass conservation [7,8] and is of paramount importance in discontinuity capturing computations of shock waves for instance where conservation must be preserved if one were to recover correct shock propagation speeds [9,10]. However, conventional finite volume methods employ one-dimensional sweeps for numerical approximation and suffer from grid orientation induced anisotropic errors (see Section 2). A reduction of the anisotropic errors in the conventional finite volume method would help achieve superior directional accuracy while simultaneously preserving discrete conservation. The fact that finite volume methods utilize a combination of reconstruction, flux computation and quadrature rules for spatial discretization makes the task of analysis of the anisotropic errors and their minimization particularly challenging and more so in presence of nonlinearity and spatial variations (see Section 4).

The present work on multidimensional conservative finite volume method with minimal anisotropic errors belongs to the larger class of physics-compatible methods. Such methods aim at enforcing essential physical properties of the continuum model in the discrete approximation of the governing equations through appropriate changes in the discretization methodology. Conservative finite difference and finite volume methods [2,33–41] (see [42–44] for an overview), symmetry-preserving methods for simulation of single [45,46] and multiphase flows [47], mimetic or compatible discretization methods [48,49] (see [50] for a review), invariant discretization methods (cf. [51,52]) and energy conserving temporal integrators (e.g. [53]) are a few immediate examples of such methods which have been shown to provide substantial improvement in accuracy over the conventional approximation methods. The importance of such methods has been recognized in a recent special issue on physics-compatible numerical methods [54]. Here, in addition to conservation we emphasize on rotational invariance as an essential property in the multidimensional finite volume method and highlight its importance through a series of test cases which clearly demonstrate the remarkable improvement in directional accuracy over the conventional conservative finite volume method. The development and realization of two and three-dimensional finite volume schemes which