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Axisymmetric fully spectral code for hyperbolic equations

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ABSTRACT

We present a fully pseudo-spectral scheme to solve axisymmetric hyperbolic equations of second order. With the Chebyshev polynomials as basis functions, the numerical grid is based on the Lobbato (for two spatial directions) and Radau (for the time direction) collocation points. The method solves two issues of previous algorithms which were restricted to one spatial dimension, namely, (i) the inversion of a dense matrix and (ii) the acquisition of a sufficiently good initial-guess for non-linear systems of equations. For the first issue, we use the iterative bi-conjugate gradient stabilized method, which we equip with a pre-conditioner based on a singly diagonally implicit Runge–Kutta ("SDIRK"-) method. In this paper, the SDIRK-method is also used to solve issue (ii). The numerical solutions are correct up to machine precision and we do not observe any restriction concerning the time step in comparison with the spatial resolution. As an application, we solve general-relativistic wave equations on a black-hole space–time in so-called hyperboloidal slices and reproduce some recent results available in the literature.

1. Introduction

There is no need to emphasize the central role that partial differential equations (PDEs) play in physics, since probably most processes in nature can be modeled by them. While the mathematical theory of linear PDEs is a well established field and exact solutions are available in many cases, treating a system of nonlinear PDEs analytically is a much more complicated task, where rarely does one get an exact solution. In such cases, numerical simulation is a very useful tool to approach the problem. However, the numerical solution to the PDEs system is not exact. Rather, it is beset with errors introduced by the chosen numerical method.

Even though there is a wide range of numerical methods to treat PDEs, our interests lie in those which render highly accurate solutions, ideally close to machine precision. In this context, (pseudo-)spectral methods are probably the best choice, as they have the remarkable capability of providing exponential convergence rate when the underlying problem admits a regular solution [1,2].

Spectral methods have been widely used for systems of *elliptic* PDEs [1,2]. For time-dependent problems though, usually the spatial and time directions are treated differently (see e.g. [3] for a discussion on the use of spectral methods for time-dependent problems). Typically, spectral methods are restricted to the spatial grid, while the time evolution is performed with a standard time integrator (for instance, the well known explicit 4th order Runge–Kutta scheme). Apart from the loss of the solution's accuracy, in many cases such combined methods have to satisfy the so-called Courant–Friedrichs–Lewy condition (CFL condition) which imposes a restriction on the time step according to the spatial grid size. A way to overcome these caveats is to apply a spectral method to *all* directions, i.e. to both space *and* time. To the best of our knowledge, a first

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study along this line was performed for *parabolic* PDEs [4], while a fully spectral code for hyperbolic equations was first proposed in [5]. These works were restricted to problems with one spatial and one time dimension. Recently, a first step towards a fully spectral code in higher dimensions was presented in [6] for advection equations.

In this work, we extend the results of [5,7,8] and introduce a fully spectral code for axisymmetric hyperbolic PDEs. In particular, we address two limitations of the code presented in [8]. The first issue concerns the inversion of a large dense matrix, which is known to be a delicate aspect inside a fully spectral solver (see e.g. [9]). In higher dimensions (for our axisymmetric code: two spatial dimensions and one time dimension, henceforth "$2+1$"), the computational costs of a direct method (such as LU-decomposition) become prohibitive (as reported, for instance, in [6]). The second issue arises since the treatment of nonlinear equations by means of the Newton–Raphson scheme (which is frequently used in the realm of spectral methods) requires an adequate initial-guess for the solution. The examples shown in [8] are small perturbations of an explicitly known solution. For stronger perturbations, the method again becomes expensive, as one would have to go through many intermediate steps with gradually increasing perturbation parameter until the desired situation is reached. Even more seriously, in cases in which no corresponding approximate solution can be identified, the method might not work at all, as no initial-guess would be available.

In order to solve the first issue, we invert the dense matrices appearing within the spectral code in terms of the iterative "bi-conjugate gradient stabilized method" (BiCGStab) [10]. In particular, we endow the BiCGStab method with a pre-conditioner based on a "singly diagonally implicit Runge–Kutta method" (SDIRK), which reduces significantly the number of iterations needed for convergence inside the BiCGStab-method. The SDIRK-method is also the algorithm of our choice to solve the second issue.[1]

Note that a delicate issue arises when the spatial domain is unbounded (for instance, with a radius coordinate $r \in [0,\infty)$). In principle, one could introduce artificial boundaries at large, but finite radii on the numerical grid. However, in order to complete the mathematical problem, one needs extra conditions which are compatible with the differential equations and with the underlying physical scenario. Unfortunately, it is a very difficult task to fully control the influence from such artificial boundaries on the numerical solution. Especially, pseudo-spectral methods are particularly sensitive to this feature, and the simulation is bound to break down after a few time steps, when errors originating from the boundary accumulate.

For such unbounded domains, a common strategy is the compactification of the spatial domain via the introduction of a suitable coordinate system. This approach is not just a clever trick to treat the equations numerically, but it is an active line of research within General Relativity. In [11], Penrose introduced the concept of conformal infinity, which brings so-called future null infinity $\mathscr{I}^+$ (which is the set of points which are approached asymptotically by light rays and gravitational waves) to a finite coordinate value. In this manner, the conformal concept permits the inclusion of this surface in the numerical grid and it removes the necessity of imposing artificially boundary conditions. Generally speaking, the physical space–time metric $g_{\mu\nu}$ is rescaled by a conformal factor Ω and one works with the regular *conformal* metric $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ (see [12] for a review). In the compactified "unphysical" space–time endowed with the conformal metric, future null infinity is described simply by the hypersurface $\Omega_{|\mathscr{I}^+} = 0$. In this context, space-like surfaces extending up to $\mathscr{I}^+$ are referred to as hyperboloidal slices.

From the physical point of view, the inclusion of $\mathscr{I}^+$ in the numerical grid allows one to precisely extract the gravitational wave content of the space–time. However, the conformal approach has the drawback of introducing singular terms ($\sim \Omega^{-n}$ for some integer n) into the field equations. For relativistic gravitational fields, Friedrich [13] reformulated the Einstein equations in terms of the conformal metric in such a way that they are manifestly regular at $\mathscr{I}^+$. Regarding the practical purpose, the system of equations turned out to be rather complicated for the numerical treatment (but see [14], for a complete discussion on the theoretical and numerical development of this formalism). Recently, Moncrief and Rinne [15] showed that the apparently singular boundary terms appearing in the hyperboloidal concept can be explicitly evaluated at $\mathscr{I}^+$. Their scheme is based on a constrained and gauge fixing formulation of the field equation and, as a result, stable dynamical numerical evolutions were presented in [16,17]. An alternative approach was suggested in [18] where a free unconstrained evolution of the field equations in terms of the generalized harmonic gauge [19] is described.

In a future research effort we intend to numerically solve the Einstein equations in such a harmonic formalism, in which they form a system of wave equations. The work presented in this article is a pre-intermediate step toward this goal and we show, in particular, that our fully spectral method can handle equations with singular terms and that it is suitable to deal with the conformal approach. Another feature of our code is the capability of solving problems with *free* boundaries such as the dynamics of a spherically symmetric, oscillating, self-gravitating star whose spatial location of its surface is unknown at the outset but needs to be determined simultaneously with the evolution of the gravitational field and the star's density and velocity fields.

The paper is organized as follows. Section 2 introduces the numerical scheme. It describes both the fully spectral and the SDIRK methods and it ends with a simple example that illustrates the application of the SDIRK method in a free-boundary problem. In Section 3 we apply the scheme to two different problems. At first we reproduce the results of [8] regarding the computation of the dynamics of a spherically symmetric, oscillating, self-gravitating star. We focus, in particular, on

[1] In this case, the procedure can be seen as follows: given initial data and boundary conditions, we first solve the PDE system with an implicit Runge–Kutta integrator in time and spectral methods for spatial coordinates; this first solution is then refined by the use of spectral methods in time. We mention however that (i) for linear equations such a preliminary step is not necessary and (ii) for non-linear equations, the initial guess could alternatively be determined by other means.