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A new Green's function Monte Carlo algorithm for the solution of the two-dimensional nonlinear Poisson–Boltzmann equation: Application to the modeling of the communication breakdown problem in space vehicles during re-entry

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ABSTRACT

The objective of this paper is the exposition of a recently-developed, novel Green's function Monte Carlo (GFMC) algorithm for the solution of nonlinear partial differential equations and its application to the modeling of the plasma sheath region around a cylindrical conducting object, carrying a potential and moving at low speeds through an otherwise neutral medium. The plasma sheath is modeled in equilibrium through the GFMC solution of the nonlinear Poisson–Boltzmann (NPB) equation. The traditional Monte Carlo based approaches for the solution of nonlinear equations are iterative in nature, involving branching stochastic processes which are used to calculate linear functionals of the solution of nonlinear integral equations. Over the last several years, one of the authors of this paper, K. Chatterjee has been developing a philosophically-different approach, where the linearization of the equation of interest is not required and hence there is no need for iteration and the simulation of branching processes. Instead, an approximate expression for the Green's function is obtained using perturbation theory, which is used to formulate the random walk equations within the problem sub-domains where the random walker makes its walks. However, as a trade-off, the dimensions of these sub-domains have to be restricted by the limitations imposed by perturbation theory. The greatest advantage of this approach is the ease and simplicity of parallelization stemming from the lack of the need for iteration, as a result of which the parallelization procedure is identical to the parallelization procedure for the GFMC solution of a linear problem. The application area of interest is in the modeling of the communication breakdown problem during a space vehicle's re-entry into the atmosphere. However, additional application areas are being explored in the modeling of electromagnetic propagation through the atmosphere/ionosphere in UHF/GPS applications.

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1. Introduction

The electric force arising out of even a small percentage imbalance of charged particles within plasma environments is very strong. Hence, any excess charge is quickly eliminated by particle motion and quasi-neutrality is usually a good approximation in relatively high-density, low-temperature plasmas. However, quasi-neutrality does not prevail for ionized gases in the vicinity of a solid surface and this non-neutral region is called a plasma sheath [1]. In the next section, we provide a simple model for a plasma sheath around a long cylindrical object carrying a potential moving at low speeds in neutral medium and derive approximate expressions for the potential in the vicinity of the sheath.

1.1. Electrostatic potential in a plasma sheath

The simplest model for a plasma sheath surrounding a conducting object is the Debye potential for a conducting sphere within a plasma region [1–3]. In this work, we adapt this model to problems in cylindrical geometry. The region of interest is an infinite domain of neutral medium with a number density of n_0, capable of being ionized into electrons and singly charged positive ions. An infinitely long conducting cylinder of radius R carrying a potential φ_s is traveling through this region at low, non-relativistic speeds. Assuming that the number density of the charged particles surrounding the moving cylinder follows the Boltzmann distribution as a function of the potential φ, the number densities of ions (n_i) and electrons (n_e) can be written as

$$n_i(\varphi) = n_0 e^{-\frac{q\varphi}{k_B T_i}},$$
$$n_e(\varphi) = n_0 e^{+\frac{q\varphi}{k_B T_e}}, \tag{1}$$

with q being the magnitude of the electronic charge, ε_0 being the permittivity of free space, k_B being the Boltzmann constant, and T_i and T_e being the ion and electron temperatures respectively. The electric potential φ can then be obtained from the Poisson's equation

$$\nabla^2\varphi = -\frac{\rho}{\varepsilon_0} = -\frac{q}{\varepsilon_0}\big[n_i(\varphi) - n_e(\varphi)\big], \tag{2}$$

and because of the nonlinear potential dependent concentration term on the right side of Eq. (2), this equation is known as the nonlinear Poisson–Boltzmann (NPB) equation [1–3]. The linearization of Eq. (2) leads to

$$\nabla^2\varphi - \frac{1}{\lambda_D^2}\varphi = 0, \tag{3}$$

where

$$\frac{1}{\lambda_D^2} = \frac{q^2 n_0}{\varepsilon_0 k_B}\left(\frac{1}{T_e} + \frac{1}{T_i}\right), \tag{4}$$

with λ_D being referred to as the plasma Debye length. Based on the linearized equation, we define an equivalent temperature given by

$$T_{eq} = \frac{T_e T_i}{T_e + T_i} \tag{5}$$

and the Debye length can be expressed in terms of this equivalent temperature as

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k_B T_{eq}}{q^2 n_0}}. \tag{6}$$

For an infinite cylinder, the potential φ is dependent only on the radial coordinate and given by the solution of the equation

$$r^2\frac{d^2\varphi}{dr^2} + r\frac{d\varphi}{dr} - r^2\frac{1}{\lambda_D^2}\varphi = 0. \tag{7}$$

The equation above is the modified Bessel equation of order zero where the modified Bessel equation of order ν is given by

$$r^2\frac{d^2\varphi}{dr^2} + r\frac{d\varphi}{dr} - \big(r^2 + \nu^2\big)\varphi = 0. \tag{8}$$

As a result, the solution to Eq. (7) is given by

$$\varphi(r) = A I_0\left(\frac{r}{\lambda_D}\right) + B K_0\left(\frac{r}{\lambda_D}\right), \tag{9}$$