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Artificial boundary conditions and finite difference approximations for a time-fractional diffusion-wave equation on a two-dimensional unbounded spatial domain

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A B S T R A C T

We consider the numerical solution of the time-fractional diffusion-wave equation on a two-dimensional unbounded spatial domain. Introduce an artificial boundary and find the exact and approximate artificial boundary conditions for the given problem, which lead to a bounded computational domain. Using the exact or approximating boundary conditions on the artificial boundary, the original problem is reduced to an initial–boundary-value problem on the bounded computational domain which is respectively equivalent to or approximates the original problem. A finite difference method is used to solve the reduced problems on the bounded computational domain. The numerical results demonstrate that the method given in this paper is effective and feasible.

1. Introduction

The time-fractional diffusion-wave equation is a mathematical model of a wide range of important physical phenomena, including ordinary diffusion, dispersive anomalous diffusion [14,42,54], Pipkin's viscoelasticity [27,44,45,37], colloid, glassy and porous materials, in fractals, percolation clusters [21,50], biological systems [36], random and disordered media [40,41], finance [39], quantum mechanics [20], and constant-Q seismic-wave propagation [3,4,38].

Many authors have tried to construct analytical solutions to problems of time-fractional differential equations. This has been done for example in [47,51] for the time-fractional diffusion-wave equations. Here, the corresponding Green's functions and their properties are obtained in terms of Fox functions. In [15,16] the similarity method and Laplace transform techniques are used to obtain the scale-invariant solution of time-fractional diffusion-wave equations in terms of the Wright function. The solutions to the time-fractional advection–dispersion equation in the whole space and the half space are given in [22] by resorting to Fourier–Laplace transforms. Non-central-symmetric solutions in a sphere are constructed by using the Laplace transform for time and the Legendre transform, the finite Fourier transform and the finite Hankel transform in space in [46]. Advances in obtaining analytical solutions for the multi-term time-fractional diffusion-wave/diffusion equations can be found in [23,24,52].

However, it is usually not possible to obtain the analytic solution for the general case. Instead, Liu et al. [32] use a first-order finite difference scheme in both time and space direction, and some stability conditions are derived. In [49] a

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finite difference scheme for the fractional diffusion-wave equation is proposed. A compact difference scheme for solving the fractional diffusion-wave equation is presented in [9], and a numerical method using the decomposition method for this equation is considered in [1], while the implicit numerical scheme for a fractional diffusion equation is considered in [28]. The paper [7] describes and analyzes the time-discretization of time-fractional diffusion-wave equations by convolution quadrature. The backward Euler scheme is used to discretize the first-order time derivative and the L^1 scheme is used to approximate the fractional-order time derivative. The implicit difference approximation for the two-dimensional space–time fractional diffusion equation is considered in [55]. A finite difference scheme in time and the Legendre spectral method in space for the time-fractional diffusion equation are employed in [30] and a convergence rate of $(2-\alpha)$-order in time and spectral accuracy in space of the method was rigorously proved. Then in [29], a space–time spectral method for the time fractional diffusion equation is presented and the well-posedness of this method by introducing a well-suited variational formulation is established. The spectral accuracy of the method is proven by providing a priori error estimate. A conservative difference approximation for the time-fractional diffusion equation is proposed and analyzed in detail in [48]. Other numerical approaches to the time-fractional diffusion equation can be found in [2,5,6,31,33,34,43,53,56] and the references therein.

In this paper, we consider the numerical solution of the time-fractional diffusion-wave equation in a two-dimensional unbounded domain. The unboundedness of the domain is one essential difficulty for finding the numerical solution of the given problems. The artificial boundary method [10,11,17–19] is a powerful tool for the numerical solution of initial–boundary-value problems on unbounded domains. By introducing an artificial boundary, the given domain is divided into two parts, a finite computational domain and an infinite domain. A suitable boundary condition is imposed on the artificial boundary, such that the solution of the problem with the suitable boundary condition on the artificial boundary in the finite computational domain is a good approximation of the original problem.

In [13,12], the exact artificial boundary condition (ABC) for the one-dimensional time-fractional sub-diffusion equation $(0<\alpha<1)$ is derived by resorting to Laplace transform techniques. Then the finite difference methods are used to solve the reduced problem on bounded domain. The convergence rate and stability are also established in these two papers. Similar techniques for deriving the absorbing boundary conditions of Engquist and Majda are used in [8] to obtain the absorbing boundary condition for the two-dimensional time-fractional wave equation with $1<\alpha<2$. In the present paper, we derive the exact and a series of approximate artificial boundary conditions for the time-fractional diffusion-wave equations $(0<\alpha<2)$ in a two-dimensional unbounded spatial domain.

The paper is organized as follows. In Section 2, the exact boundary condition is derived on the given artificial boundary Γ for the time-fractional diffusion-wave equation on an unbounded two-dimensional spatial domain; that is, the relationship between $\frac{\partial u}{\partial n}|_\Gamma$ and $\frac{\partial u}{\partial t}|_\Gamma$ is established. Moreover, a series of artificial boundary conditions with high accuracy is obtained. By means of the artificial boundary conditions, a family of approximate problems for the original problem on the bounded computational domain is constructed. Section 3 contains the stability estimates for the two finite difference schemes, and in Section 4 we present the analysis of the order of convergence. To demonstrate the accuracy and efficiency of our ABCs, numerical examples are given in Section 5. Finally, we add some concluding remarks in Section 6.

2. The artificial boundary conditions (ABCs)

In this section, we consider the artificial boundary conditions for the following time-fractional diffusion-wave equations:

$$\begin{cases} {}^cD_t^\alpha u(x,t)=\Delta u(x,t)+f(x), & x\in\Omega,\ t\in(0,T],\ 0<\alpha<1,\\ u(x,0)=\phi(x), & x\in\Omega,\\ u(x,t)|_{\partial\Omega}=0, & t\in(0,T], \end{cases} \tag{2.1a}$$

$$\begin{cases} {}^cD_t^\alpha u(x,t)=\Delta u(x,t)+f(x), & x\in\Omega,\ t\in(0,T],\ 1<\alpha<2,\\ u(x,0)=\phi(x), & x\in\Omega,\\ u_t(x,0)=\varphi(x), & x\in\Omega,\\ u(x,t)|_{\partial\Omega}=0, & t\in(0,T], \end{cases} \tag{2.1b}$$

where Ω is an unbounded domain in $\mathbb{R}^2$ given by

$$\Omega:=\big\{(x_1,x_2)\mid -\infty<x_1\le 0,\ 0<x_2<a;\ 0\le x_1<+\infty,\ 0<x_2<b\big\}, \tag{2.2}$$

with $b\ge a>0$. Here, $f(x)$, $\phi(x)$, $\varphi(x)$ are given functions on $\overline{\Omega}$ whose supports are compact, and ${}^cD_t^\alpha$ denotes the Caputo fractional derivative of order α with respect to t; it is defined by

$${}^cD_t^\alpha u(x,t):=\begin{cases} \frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{1}{(t-s)^\alpha}\frac{\partial u(x,s)}{\partial s}ds, & 0<\alpha<1,\\ \frac{1}{\Gamma(2-\alpha)}\int_0^t\frac{1}{(t-s)^{\alpha-1}}\frac{\partial^2 u(x,s)}{\partial s^2}ds, & 1<\alpha<2. \end{cases} \tag{2.3}$$

$\phi(x)$, $\varphi(x)$ satisfy compatibility conditions

$$\phi(x)|_{\partial\Omega}=0,$$
$$\varphi(x)|_{\partial\Omega}=0. \tag{2.4}$$