Journal of Computational Physics 276 (2014) 729–739

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Water wave propagation in unbounded domains. Part I: Nonreflecting boundaries ☆

G.I. Jennings, S. Karni *, J.B. Rauch

Department of Mathematics, University of Michigan, United States

ARTICLE INFO

Article history:
Received 2 November 2012
Received in revised form 10 February 2014
Accepted 21 February 2014
Available online 28 February 2014

Keywords:
Nonreflecting boundary conditions
Fluid dynamics
Incompressible flow
Water waves

ABSTRACT

This paper concerns non-reflecting boundary conditions for linearized 2D incompressible, irrotational, inviscid free surface flows in deep water. The linearization at flat water of the water wave equation is factored as a product of one-way equations. The one-way factors involve a nonlocal half derivative in space. They serve to construct essentially non-reflective and absorbing layers. The one way nature of the equation prevents errors at the boundary from propagating back and polluting the solution in the interior of the domain. Two versions of the one-way equations are discussed, with and without additional wave damping. Numerical results are presented.

1. Introduction

Water waves are often studied in unbounded domains. For example, when modeling flow around a vessel on an open sea, effects from the distant ocean bottom or shorelines are negligible compared to the surface wave-body interactions. To be able to compute such solutions, the domain first needs to be truncated to a finite size. This is done by introducing an artificial boundary at some arbitrary, ideally not too large, distance. Boundary conditions are then needed on the artificial boundary, that mimic the unbounded domain and ideally make the boundary invisible to outgoing waves. This type of boundary conditions is called absorbing or non-reflecting boundary conditions, and are often nontrivial to formulate. One may view non-reflecting boundary conditions as compatibility conditions. In general, to make the boundaries transparent to outgoing waves, one first needs to figure out the structure of those waves, then prescribe conditions that are compatible with that structure. The details depend on the context. In one dimensional flow, the task is simpler, at least in principle. Waves move either left or right, either towards or away from the boundary. In multi-dimensional flows, waves may approach the boundary from infinitely many directions, and absorbing all of then via a small number of boundary statements is often difficult, impossible, or impractical. One then settles for less than perfect absorption of outgoing waves (see, for example, [1,2]).

The work in this paper falls somewhere between a one- and two-dimensional flow problem. It concerns two-dimensional incompressible, irrotational, inviscid free surface flow in deep water, but the problem is reduced to solving an equation which is one-dimensional in space. In [7], Wu proposes a formulation of the equations that enables establishing finite-time existence and uniqueness of solutions. It proceeds by mapping the infinitely deep two-dimensional domain into the lower half plane, and formulates the problem in terms of the horizontal flow velocity on the free surface.

In this work, we construct absorbing boundary conditions for the linearization at zero of the water wave equation. We use Fourier analysis to identify the structure of left and right moving water waves and derive a one-way version of the

☆ The work was supported in part by NSF, Awards # DMS 0609766 and # DMS 0807600, and by a University of Michigan Elizabeth Crosby Award.
* Corresponding author.

http://dx.doi.org/10.1016/j.jcp.2014.02.032
0021-9991/

Fig. 1. The domain of the fluid for the system of equations in (1).

equation, which we implement as an absorbing layer near the artificial computational boundary. The one-way equation supports the propagation of water waves in one direction. It is a *fractional partial differential equation* involving a nonlocal operator corresponding to half a derivative in space. The one way nature of the equation prevents potential errors at the boundary from propagating back and polluting the solution in the interior of the domain. We demonstrate the efficiency of the one-way water wave equation in absorbing outgoing waves. Two versions of the one-way equations are presented, with and without additional wave damping. The present paper forms Part I of a two-part report, and focuses on the derivation of the one-sided equation used at non-reflecting boundaries. Part II [4] focuses on numerics, and presents efficient numerical methods for the one-way WWE (15).

This paper is organized as follows: In Sections 2 and 3, we briefly describe the full water wave equation, discuss its properties, and illustrate some solutions. In Section 4, we derive a one-way version of the equation and analyze its properties. In Section 5, we implement the one-way equation in an outer layer near the artificial boundary, and demonstrate its effectiveness in absorbing outgoing waves.

2. The water wave equation

Consider the Euler equations for two-dimensional incompressible irrotational free surface flow:

$$\begin{aligned} &\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} = -(0,1) - \nabla p \\ &\nabla\cdot\mathbf{v} = 0 \\ &\nabla\times\mathbf{v} = 0. \end{aligned} \tag{1}$$

Here, $\mathbf{v}$ denotes the fluid velocity and p is the fluid pressure, the fluid is of density $\rho = 1$, and the gravitational field given by $(0, -1)$. The domain is assumed to be infinitely deep, the top boundary is non-self-intersecting and approaches the horizontal axis at $\pm\infty$ with standard boundary conditions: (i) $p = 0$ on the free surface; (ii) The free surface is a streamline; and (iii) $|\mathbf{v}(x, y, t)| \to 0$ as $|(x, y)| \to \infty$. The problem set-up is shown in Fig. 1.

In [7], Wu uses conformal mapping to transform the fluid domain to the lower half plane at each time t, and obtains an equation for the horizontal velocity on the free surface. The vertical velocity on the free surface is related to the horizontal velocity via a Hilbert transform, and the velocity field below the free surface can be recovered by solving Laplace's equation in $\mathbf{v}$ with appropriate boundary conditions on the free surface. The reader is referred to [3,7] for the details. In this paper, we are concerned only with the horizontal velocity on the free surface. It can be shown that the horizontal velocity on the free surface satisfies an equation of the form

$$u_{tt} + |D|u = g(u, \tilde{x}, t). \tag{2}$$

Here, $\tilde{x}$ represents a parameterization of the free surface, originating from the domain mapping. For convenience, we drop the $\tilde{}$ notation, and from now on we write $u = u(x, t)$, where x is related but not identical to the physical x-coordinate.

The function g can be shown to be at least quadratic in u, so that for small u, $g = \mathcal{O}(u^2)$. In particular, $u = 0$ is an exact solution, and neglecting g yields the linearization at that solution.

3. Fourier transforms and the linearization of the WWE

The operator $|D|$ in (2) is defined using the Fourier transform and its inverse,

$$\widehat{f}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,dx, \qquad f(x) := \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi i x\xi}\,d\xi.$$

Recall that $\partial_x = iD$ is the usual derivative operator, so that

$$\widehat{Df}(\xi) = 2\pi\xi\widehat{f}(\xi), \qquad \widehat{\partial_x f}(\xi) = 2\pi i\xi\widehat{f}(\xi).$$

In analogy, the expression $|D|$ denotes the operator defined by

$$\widehat{|D|f}(\xi) = |2\pi\xi|\widehat{f}(\xi). \tag{3}$$