Journal of Computational Physics 261 (2014) 230–243

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

An asymptotic solution approach for elliptic equations with discontinuous coefficients

M. Oevermann [a,*], R. Klein [b]

[a] Chalmers University of Technology, Department of Applied Mechanics, Gothenburg, Sweden
[b] Freie Universität Berlin, Fachbereich Mathematik & Informatik, Berlin, Germany

ARTICLE INFO

Article history:
Received 21 August 2013
Received in revised form 4 December 2013
Accepted 5 December 2013
Available online 3 January 2014

Keywords:
Elliptic equations
Embedded interface
Variable and discontinuous coefficients
Two-phase flow

ABSTRACT

When the coefficients of an elliptic problem have jumps of several orders of magnitude across an embedded interface, many iterative solvers exhibit deteriorated convergence properties or a loss of efficiency and it is difficult to achieve high solution accuracies in the whole domain. In this paper we present an asymptotic solution approach for the elliptic problem $\nabla \cdot (\beta(\boldsymbol{x})\nabla u(\boldsymbol{x})) = f(\boldsymbol{x})$ on a domain $\Omega = \Omega^+ \cup \Omega^-$ with piecewise constant coefficients β^+, β^- with $\beta^+ \gg \beta^-$ and prescribed jump conditions at an embedded interface Γ separating the domains Ω^+ and Ω^-. We are in particular focusing on a problem related to fluid mechanics, namely incompressible two-phase flow with a large density ratio across the phase boundary, where an accurate solution of the velocity depends on the accurate solution of a pressure Poisson equation with equal local relative errors in the whole domain. Instead of solving the equation in a single solution step we decompose the problem into two consecutive problems based on an asymptotic analysis of the physical problem where each problem is asymptotically independent of the ratio of coefficients $\varepsilon = \beta^-/\beta^+$. The proposed methods lead to a robust and accurate solution of the elliptic problem using standard black-box iterative solvers.

1. Introduction

We are looking for the solution of the elliptic problem

$$\nabla \cdot \big(\beta(\boldsymbol{x})\nabla u(\boldsymbol{x})\big) = f \tag{1}$$

in the domain $\Omega \subseteq \mathbb{R}^2$ with boundary condition

$$u(\boldsymbol{x}) = u_{\mathcal{B}}(\boldsymbol{x}) \quad \text{or} \quad u_{\boldsymbol{n}}(\boldsymbol{x}) = \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}) = u_{\boldsymbol{n},\mathcal{B}}(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \partial\Omega. \tag{2}$$

Here, $u_{\mathcal{B}}(\boldsymbol{x})$ denotes a Dirichlet boundary conditions and $u_{\boldsymbol{n},\mathcal{B}}(\boldsymbol{x})$ a Neumann boundary condition where $\boldsymbol{n}$ is the outer unit normal to the boundary $\partial\Omega$. The coefficient $\beta(\boldsymbol{x})$ is assumed to be piecewise constant:

$$\beta(\boldsymbol{x}) = \begin{cases} \beta^+ & \text{for } \boldsymbol{x} \in \Omega^+, \\ \beta^- & \text{for } \boldsymbol{x} \in \Omega^-, \end{cases} \tag{3}$$

* Corresponding author.
E-mail addresses: michael.oevermann@chalmers.se (M. Oevermann), rupert.klein@math.fu-berlin.de (R. Klein).

0021-9991/$ – see front matter
http://dx.doi.org/10.1016/j.jcp.2013.12.027

where Ω^+ and Ω^- form a decomposition of the domain Ω into non-overlapping sub-domains separated by an interface Γ. Along the interface we prescribe jump conditions for the solution and its derivative in the normal direction:

$$[\![u]\!]_\Gamma = u^+(\boldsymbol{x}) - u^-(\boldsymbol{x}) = g(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Gamma, \tag{4}$$

$$[\![\beta u_n]\!]_\Gamma = \beta^+ u_{\boldsymbol{n}}^+(\boldsymbol{x}) - \beta^- u_{\boldsymbol{n}}^-(\boldsymbol{x}) = h(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Gamma, \tag{5}$$

with $u_{\boldsymbol{n}} = (\nabla u \cdot \boldsymbol{n})$ and the unit normal vector on Γ denoted by $\boldsymbol{n}$.

Variable coefficient elliptic equations with discontinuous coefficients across interfaces are ubiquitous in many branches of scientific and engineering applications, e.g., electrostatics, fluid mechanics, heat conduction, diffusion problems, elasticity, etc. Numerous different discretization methods for this type of problem have been proposed in the past, many of them related to problems of two-phase flow. Examples are conforming [1,4,9] and non-conforming finite element methods [20,12,13], finite difference methods [26,15,16,22,10,29,19], and finite volume methods [21,23].

It is a known problem that many iterative solvers for systems of linear equations exhibit convergence problems and/or a loss of efficiency for interface problems with discontinuous coefficients of different orders of magnitude, e.g. 1:1000 for the densities of air and water or up to 1:10 000 for the diffusion coefficients in gas–liquid systems (for interface diffusion problems in gas–solid systems, e.g. case hardening of steel via introduction of carbon by diffusion (carburising) into the local surface, this ratio can be even higher). The poor convergence is often attributed to the existence of a few very low eigenvalues in the spectrum of even preconditioned systems leading to high condition numbers, see [28,8].

Several methods have been proposed to obtain robust and efficient iterative solvers for elliptic problems with highly discontinuous coefficients. Among these methods are domain decomposition methods and preconditioners (e.g. Schwarz methods [5,17,6,7], the balancing domain decomposition method [18], and the finite element tearing and interconnecting (FETI) method [14]), Schur complement preconditioners [24,25], and multigrid algorithms [3].

The above mentioned iterative methods have shown good convergence properties for elliptic problems with discontinuous coefficients – at least for problems with relatively simple interface topologies. However, our own experience using multigrid preconditioned Krylov type methods available from public domain software packages showed deteriorated or no convergence at all for certain interface topologies and a high ratio of the coefficients (usually higher than 100). Furthermore, most of the specialized preconditioners are not available in standard black-box solvers or public domain packages for the solution of large sparse systems of linear equations and are therefore not easily available.

The cited work on iterative solvers mostly focuses on the efficiency of the iterative solver only, i.e. the number of iterations, but not on the accuracy of the obtained solution for a specific physical problem. Here we propose a quite different approach to the problem: We focus on analysing the structure of the underlying physical problem and develop a robust and accurate method which can be solved by standard black box solvers. The method will follow the lines of the asymptotic analysis of the Navier–Stokes equations in the limit of vanishing density ratio $\varepsilon = \rho_1/\rho_2$ as presented in [22]. Our asymptotic approach has some similarities with domain decomposition methods in the sense that it leads to two decoupled problems where the interface defines the boundary for the leading order solution. It could be considered to be an asymptotics-induced preconditioner. However, in contrast to domain decomposition methods we do not rely on an iterative process requiring (optimized) domain-coupling boundary conditions, but instead require only a standard discretization for Dirichlet boundary conditions and any type of sharp-interface discretization which is capable of dealing with embedded interfaces and interface jump conditions, e.g. immersed interface methods, ghost fluid methods, extended finite element methods, and finite volume methods.

The paper is organized as follows. In Section 2 we will shortly motivate our specific interest in investigating (1), especially the scaling of the right hand side of (1). In Section 3 we provide an asymptotic analysis of the elliptic problem for the physical problem of interest as discussed in Section 2 and present our proposed two-step solution scheme for solving (1). In Section 4 we show results and compare the two-step asymptotic approach with a standard single-step approach. Some conclusions are drawn in Section 5.

2. Incompressible two-phase flow

Our motivation for investigating (1) are the incompressible, isothermal Navier–Stokes equations for two-phase flow with surface tension. In each fluid phase (A) or (B) the momentum and mass-balance equations are

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu \Delta \boldsymbol{u}, \tag{6}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{7}$$

Here $\boldsymbol{u}$ is the velocity, ρ the density, p the dynamic pressure, and ν denotes the kinematic viscosity. The boundary conditions at the interface between two immiscible viscous fluids are the continuity of the velocity

$$[\![\boldsymbol{u}]\!] = 0, \tag{8}$$

and the dynamic boundary condition balancing the normal and tangential stresses: