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Compact cell-centered discretization stencils at fine–coarse block structured grid interfaces

Alexander Pletzer *, Ben Jamroz, Robert Crockett, Scott Sides

Tech-X Corp., 5621 Arapahoe Ave., Boulder, CO 80303, USA

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ABSTRACT

Different strategies for coupling fine–coarse grid patches are explored in the context of the adaptive mesh refinement (AMR) method. We show that applying linear interpolation to fill in the fine grid ghost values can produce a finite volume stencil of comparable accuracy to quadratic interpolation provided the cell volumes are adjusted. The volume of fine cells expands whereas the volume of neighboring coarse cells contracts. The amount by which the cells contract/expand depends on whether the interface is a face, an edge, or a corner. It is shown that quadratic or better interpolation is required when the conductivity is spatially varying, anisotropic, the refinement ratio is other than two, or when the fine–coarse interface is concave.

1. Introduction

Elliptic, and in particular Laplace and Poisson type equations arise in many settings, including in reaction–diffusion problems, in incompressible computational fluid dynamics when computing the pressure, in time dependent heat conduction problems, and in orthogonal mesh generation algorithms, to name a few applications. Our focus in this article is to investigate and compare different strategies for discretizing elliptic operators in the context of the block-structured mesh refinement method [1–8].

The central idea behind block-structured mesh refinement is the partitioning of the domain into logically rectangular grid patches with different resolutions. Starting with a coarse grid, regions of the computational domain are locally refined, typically by doubling the resolution. The decision whether or not to refine can be based, for instance, on a local error estimate of the solution. At any given location, multiple levels of refinement can co-exist; the assembly of meshes forms a grid hierarchy with the coarsest mesh at the bottom and the finest mesh at the top.

One way to solve locally refined mesh problems is to apply the multigrid method [9,10], which is among the most efficient numerical algorithms available to date to solve large, multi-billion degrees of freedom, elliptic problems [11]. Several numerical libraries, e.g. Chombo [12], SAMRAI [13], FOSPACK [14], and PARAMESH [15] currently implement the multigrid algorithm. As an alternate approach, the sparse matrix representation of the operator discretized on the mesh hierarchy can be constructed and handed over to a matrix solver library such as PETSc [16]. The Chompst [17] library developed by the authors employs this approach. Chompst leverages the block structured grid data structures of the Chombo library but in contrast to Chombo, which uses a matrix-free method, Chompst builds a sparse matrix representation of the grid hierarchy.

* Corresponding author.
E-mail address: pletzer@txcorp.com (A. Pletzer).

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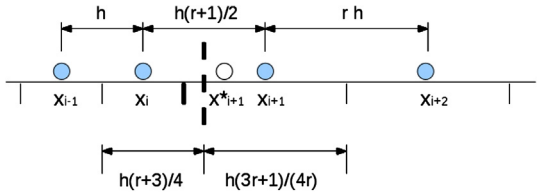

Fig. 1. The geometry of a fine–coarse grid interface in one dimension. The interface is shown as a fat vertical line, half way between x_i and x^*_{i+1}. The cell center positions are shown as solid circles. The fine ghost cell center is at distance h away from the last fine grid cell and is shown as an open circle. The linear interpolation weights at x^*_{i+1} are $\frac{r-1}{r+1}$ and $\frac{2}{r+1}$ for the fine and coarse cell center locations, respectively. The quadratic interpolation weights are $-\frac{r-1}{r+3}$, $\frac{2(r-1)}{r+1}$, $\frac{8}{(r+3)(r+1)}$ for the cell centers at $x = x_{i-1}$, x_i, and x_{i+1}, respectively.

Chompst users can then choose among a wide array of solvers available through PETSc, or if they wish, manipulate the matrix directly, e.g. in order to extract eigenvalues.

Key to assembling the sparse matrix system is the problem of accurately discretizing an elliptic operator in the vicinity of fine–coarse grid interfaces. A similar problem arises in climate modeling at the atmosphere–ocean interface [18] and in nuclear fusion modeling at the "core–edge" interface [19]. Typically, each component (e.g. atmosphere and ocean), applies its own discretization scheme on its own grid, with the grid resolution undergoing a jump at the interface. An inconsistent operator discretization or a lack of flux conservation may arise at the interface unless special care is taken.

Most reference books on multigrid do not discuss this subject in any great detail, perhaps because they consider fields to be nodal and the discretization is based on finite differences. Here, our interest lies in applying the finite volume discretization where the primary variables are located at cell centers. Although it is known since the work of Forsyth and Sammon [20] that a cell centered finite volume discretization on non-uniform grids yields the same order of accuracy as nodal finite difference discretizations, there is no published work to our knowledge that estimates the truncation and solution error of cell centered schemes for the variety of possible fine–coarse grid interfaces that can arise in two and three dimensions (flat, L-shaped, corner, etc.).

Martin and Cartwright [21] state that, for Poisson problems, quadratic interpolation is needed at fine–coarse grid interfaces in finite volume methods to achieve global second order accuracy (error is $\sim h^2$, h being the typical mesh resolution). In a finite volume discretization, where the degrees of freedom are cell centered and the fine–coarse interface runs along cell faces, a fine ghost cell is placed equidistant to the last fine cell on the coarse side of the interface (see Fig. 1). The stencil for the last fine grid cell can be derived by interpolating neighboring coarse and fine cell values at the ghost location — such a stencil was also derived by Minion [22] in the context of the refined grid MAC projection method when solving the incompressible Euler equations. Martin and Cartwright recommend the following steps to preserve second order accuracy in two or more dimensions (see Fig. 4 below). First, quadratic interpolation on the coarse grid plane, which is tangential to the interface, is applied to obtain the field value at the intersection location of the normal passing through the reference fine cell and the plane. Second, quadratic interpolation along the normal, involving both the previous intersect point and fine grid cells, is applied to populate the fine ghost value.

The goal of this article is to investigate different interpolation methods that conserve second order accuracy. To do so, we will build on the results by Johansen and Colella [23] who show that a first order truncation error at fine–coarse interfaces can be tolerated provided the fine–coarse interface forms a lower dimensional set. We will show that using first order interpolation to set the ghost value effectively amounts to charge separation in electrostatic problems and that this effect can often be cured by redistributing the cell volumes at the fine–coarse interface. Different cell volume correction factors are derived for $D-1$ dimensional (face), $D-2$ dimensional (edge), and $D-3$ dimensional (corner) interfaces. However, we also show that there are cases where *quadratic* interpolation is warranted, e.g. when the interface flux requires extrapolating coarse grid values. These correspond to the cases where the fine–coarse stencils must be shifted in Refs. [21, 23]. An important, and somewhat surprising result, is that there is no loss of accuracy resulting from extrapolating coarse grid field values.

This article is organized as follows. In Section 2 we conduct a numerical analysis of the accuracy of quadratic and linear fine ghost interpolation and compare the so-obtained finite volume discretization schemes with the finite difference approach. In Section 3 we verify the methodology developed in Section 2. We conclude by listing which cases may or may not benefit from linear interpolation in Section 4.

2. Stencils at fine–coarse interfaces

Our aim is to derive an accurate discretization of the general, second order, elliptic problem

$$L\phi \equiv \alpha\phi + \nabla\cdot(K\cdot\nabla\phi) = \rho; \quad x\in\Omega \tag{1}$$

subject to suitable boundary conditions on $\partial\Omega$. Here, ρ is a source, K a symmetric tensor, and α a scalar. The terms ρ, α, and K can be spatially varying. Depending on the problem, K may represent a thermal conductivity (in heat propagation)