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The ultra weak variational formulation of thin clamped plate problems

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ABSTRACT

We develop a new numerical scheme for a fourth order elliptic partial differential equation based on Kirchhoff's thin plate theory. In particular we extend the ultra weak variational formulation (UWVF) to thin plate problems with clamped plate boundary conditions. The UWVF uses a finite element mesh and non-polynomial basis functions. After deriving the new method we then prove L^2 norm convergence on the boundary. Finally we investigate numerically the feasibility of the UWVF for both homogeneous and inhomogeneous problems and show examples of p- and h-convergence.

1. Introduction

Dynamic plate modeling is of importance in structural analysis. While standard finite element methods can be applied to this problem, the computational burden of such methods rises rapidly at higher frequencies. The need for more efficient methods has motivated the development of several non-standard schemes using Trefftz type methods. For example, non-polynomial methods such as the variational theory of complex rays (VTCR) [1], the wave-based method (WBM) [2,3] and the discontinuous enrichment method (DEM) [4,5] have all been used to approximate thin plate problems. In order to solve dynamic plate problems (based on the Kirchhoff plate theory), we propose another non-polynomial method called the ultra weak variational formulation (UWVF) tailored to clamped plate problems. Similarly to VTCR, WBM and DEM, the UWVF has been successfully used in other wave modeling problems such as in acoustics [6–9], elasticity [10] and electromagnetism [6–8,11]. The UWVF was originally introduced by Cessenat and Després for acoustic and electromagnetic wave problems [6,7] and it belongs to the class of Trefftz type methods. The advantage of the UWVF in this case is a much simpler, and provably convergent, scheme with fewer parameter choices. However the method we shall show here is limited to the clamped plate and related boundary value problems.

In the UWVF, the computational domain is divided into geometric elements (like the finite element method). The UWVF uses non-polynomial basis functions element by element that satisfy the plate equations locally and it enforces transmission conditions between element weakly via the discontinuous Galerkin method (DGM) [11,12]. The UWVF produces sparse and block structured system matrices.

Traditionally, plane wave basis functions are used in the UWVF. In this paper we shall need to use in addition evanescent (corner) functions (decaying exponentials) in order to approximate the solution, see also [1]. In the case of the plane wave

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and evanescent (corner) function bases the integrals in the sesquilinear form can be computed analytically. It is well-known that wave based methods may suffer from ill-conditioning and the condition number needs to be carefully controlled. Therefore, we shall also report the condition numbers in our numerical test cases. Bessel basis functions can also be used in the UWVF, and this has been shown to be more stable than the plane wave basis (having smaller condition number), see [13,14] when used for the Helmholtz equation. We do not consider them here because they require more computer time to assemble the matrices for the problem.

In practice, since the UWVF is a DGM with discontinuous basis functions and the number of basis functions can be easily varied from element to element [9]. Since the basis functions can vary from element to element, it can handle jumps in the material properties between elements, see for example [9,10]. In this paper, however, we have restricted our focus to the cases in which the number of basis functions do not vary from element to element and we keep the element sizes roughly the same in the domain.

The most similar existing non-polynomial method is the DEM approach in [4,5]. These papers demonstrate the efficiency of using plane waves, but the approach is quite complicated requiring Lagrange multipliers at vertices in the mesh, and results in a mixed system. Our approach is simpler and requires no Lagrange multipliers, but it is limited to clamped plate and related boundary conditions (the DEM uses a more general sesquilinear form that includes simply supported and free plate boundary conditions).

This paper is organized as follows. First, we derive the UWVF for the elliptic fourth order partial differential equation that is generally used in thin plate problems. Our aim is to formulate the UWVF suitable for clamped plate boundary conditions. We shall outline the basic error estimates of the UWVF based on the results in [7,6,10]. Then we show numerical results, in which we first consider the homogeneous problem and investigate the p-convergence and h-convergence for decaying (corner) function model problems. Second, we investigate the homogeneous problem in which the point source is outside of the domain. Third, we consider the inhomogeneous problem with an interior volume source. Finally, we consider the thin circular clamped plate problems and compare to the exact solution derived using a Fourier expansion.

2. Biharmonic problems

In this paper we investigate the biharmonic (fourth order) problems based on Kirchhoff's thin plate theory in 2D as in [15,2,3,16]. Let $\Omega \subset \mathbb{R}^2$ be a computational domain with the boundary Γ (assumed to be a Lipschitz polygon). Let us consider the problem of finding the displacement u such that

$$\Delta^2 u - \kappa^4 u = f \quad \text{in } \Omega, \tag{1}$$

where Δ^2 is the bi-Laplacian, the plate bending wavenumber $\kappa^4 = \rho d\omega^2/D$, with the density ρ, plate thickness d, circular frequency ω, plate bending stiffness $D = Ed^3/(12(1-\nu^2))$, modulus of elasticity E, and Poisson ratio ν. The function f is the volume source term.

In the case of fourth order partial differential equations, two boundary conditions are needed in order to solve the problem. In particular, we assume that the boundary conditions can be written (cf. the impedance boundary condition in acoustics [7,9]) as follows

$$\frac{\partial \Delta u}{\partial n} - i\sigma_1 u = Q_1\left(-\frac{\partial \Delta u}{\partial n} - i\sigma_1 u\right) + g_1 \quad \text{on } \Gamma, \tag{2}$$

$$\frac{\partial u}{\partial n} - i\sigma_2 \Delta u = Q_2\left(-\frac{\partial u}{\partial n} - i\sigma_2 \Delta u\right) + g_2 \quad \text{on } \Gamma, \tag{3}$$

where n is the outward unit normal, and $\sigma_1 = \Re\{\kappa^3\} > 0$ and $\sigma_2 = \Re\{1/\kappa\} > 0$ are numerical flux parameters and are real valued ($\Re\{\cdot\}$ denotes the real part). The parameters Q_1 and Q_2 define the boundary conditions and g_1 and g_2 are the source terms on the boundary. The special forms of the boundary conditions (2) and (3) are motivated by the upcoming integration by parts formula in (19) and (20).

In particular, the idea here is to be able to consider the clamped plate boundary conditions that are given in the form

$$u = 0 \quad \text{on } \Gamma, \tag{4}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma. \tag{5}$$

To obtain these boundary conditions we can choose $Q_1 = -1$, $Q_2 = 1$ and $g_1 = g_2 = 0$.

Provided κ is not an interior eigenvalue of the bi-Laplacian, existence and uniqueness of a solution to the biharmonic wave equation with clamped plate boundary conditions is well-known. In the general case of impedance type boundary conditions (useful for modeling infinite plates) we now verify existence and uniqueness of a solution to our non-standard boundary value problem. To do this we need to use a suitable function space. For smooth functions w define the norm

$$\|w\|_X^2 = \|\Delta w\|_{L^2(\Omega)}^2 + \|w\|_{L^2(\partial\Omega)}^2 + \left\|\frac{\partial w}{\partial n}\right\|_{L^2(\partial\Omega)}^2.$$