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Geometric interpretations and spatial symmetry property of metrics in the conservative form for high-order finite-difference schemes on moving and deforming grids

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A B S T R A C T

The role of a geometric conservation law (GCL) on a finite-difference scheme is revisited for conservation laws, and the conservative forms of coordinate-transformation metrics are introduced in general dimensions. The sufficient condition of a linear high-order finite-difference scheme is arranged in detail, for which the discretized conservative coordinate-transformation metrics and Jacobian satisfy the GCL identities on three-dimensional moving and deforming grids. Subsequently, the geometric interpretation of the metrics and Jacobian discretized by a linear high-order finite-difference scheme is discussed, and only the symmetric conservative forms of the discretized metrics and Jacobian are shown to have the appropriate geometric structures. The symmetric and asymmetric conservative forms of the metrics and Jacobian are examined by the computation of an inviscid compressible fluid on highly-skewed stationary and deforming grids using sixth-order compact and fourth-order explicit central-difference schemes, respectively. The resolution of the isentropic vortex and the robustness of the computation are improved by employing symmetric conservative forms on the coordinate-transformation metrics and Jacobian that have an appropriate geometry background. An integrated conservation of conservative quantities is also attained on the deforming grid when symmetric conservative forms are adopted to the time metrics and Jacobian.

1. Introduction

A body-fitted coordinate system is often adopted to compute fluid motion around a body of arbitrary shape. In this case, the coordinate-transformation metrics required for transformation from a body-fitted coordinate system to a Cartesian coordinate system are introduced for computation of fluid motion. The governing equation is usually written in a strong conservation form of the Euler or compressible Navier–Stokes equations, and general finite-difference schemes based on a body-fitted coordinate system are used in its solution. Although the coordinate-transformation metrics analytically satisfy the freestream preservation known as the “geometric conservation law” (GCL) [23], some discretized forms of metrics violate the GCL identities using finite-difference schemes. The GCL identities are often regarded as corresponding to freestream preservation in the body-fitted coordinate system; however, they also ensure the freestream preservation in the Cartesian coordinate system. As will be described in detail later (Section 2.1), the commutativity between strong conservation forms in Cartesian and body-fitted coordinate systems is ensured by the GCL identities.

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The GCL identities comprise the "surface closure law" (SCL)[1] and the "volume conservation law" (VCL) [3,31]. Several techniques for the discretization of the metrics and Jacobian have been proposed to satisfy discretized GCL identities on moving and deforming grids. The method for the discretization of spatial metrics using a finite-difference scheme and an averaging operation was introduced by Pulliam and Steger [20]. Moreover, a method of rewriting the analytical expression for spatial metrics to their conservative forms using a second-order central-difference scheme was proposed by Thomas and Lombard [23,24]. The same discretizations were reconstructed from the viewpoint of the geometrical meanings of spatial metrics, e.g., a finite-volume scheme by Vinokur [25], and they were subsequently extended to the time metrics and Jacobian by Obayashi [18,22]. However, such special discretizations [20,25,18,22] of the metrics and Jacobian are based on a second-order discretization, and they are not suitable for higher-order computation. Accordingly, for a stationary grid, Visbal and Gaitonde experimentally confirmed that the conservative forms of spatial metrics [23] satisfy the discretized SCL identities using a linear higher-order central finite-difference scheme, which was later mathematically proven by Vinokur and Yee [26] and Deng et al. [6], independently. In this paper, we refer to the conservative forms provided by Thomas and Lombard [23] as the asymmetric conservative forms, and those with the spatial symmetry property introduced by Vinokur and Yee [26] as the symmetric conservative forms. Nonlinear high-order schemes have been addressed [16,6,9,17].

Spatial metrics are defined as the projection of area vectors onto a physical plane. This geometric interpretation of spatial metrics is the basis for the discretization of spatial metrics in terms of the finite-volume scheme. When the finite-volume scheme is employed for the discretization of spatial metrics, spatial metrics are computed as the projections of cell surfaces onto each physical plane [25]. Similarly, the Jacobian is also computed as the average of cell volumes around the grid point considered [18]. All discretization methods introduced in terms of the finite-volume scheme are based on this approximation. Recently, Deng et al. have shown that such a finite-volume like discretization (hereafter, denoted as a discretized geometric interpretation) of the spatial metrics and Jacobian is obtained in symmetric conservative forms with a second-order central-difference scheme [7], which had been also shown by Abe et al. [1,4] independently. Deng et al. have also shown that such a spatial symmetry property for the spatial metrics and Jacobian attains robust computation with high resolution using weighted compact nonlinear schemes (WCNS) on highly-skewed stationary grids. However, their discretized geometric interpretations are based on second-order central-difference and first-order boundary schemes. Therefore, the applicability of a general high-order finite-difference scheme has not been clearly shown in terms of the discretized geometric interpretation when applied to the symmetric conservative spatial metrics and Jacobian.

On the other hand, for moving and deforming grids, the method of rewriting the analytical forms of the time metrics and Jacobian to their conservative forms was proposed for a higher-order finite-difference scheme by Abe et al. [3,2]. This method is superior to the well-known method introduced by Visbal and Gaitonde [27] wherein the integrated conservation property for conservative quantities is numerically ensured with a round-off error by only using the conservative form of time metrics and Jacobian. The spatial symmetry property can be introduced to the time metrics and Jacobian in a manner similar to that for the conservative forms of spatial metrics [21]; however, the computational effect of the spatial symmetry property in time metrics has never been examined on moving and deforming grids with a high-order finite-difference scheme. In addition, with a high-order finite-difference scheme, the discretized geometric interpretation has never been investigated on the conservative forms of the time metrics.

The main purpose of this study is to obtain the geometric structures of the discretized conservative metrics and Jacobian using a high-order finite-difference scheme and to examine the computational effect of the spatial symmetry property for conservative forms of the time metrics and Jacobian on moving and deforming grids. First, the role of the GCL identities is revisited in general dimensions in Section 2, and the conservative forms of the metrics are generally introduced in Section 3.1. For the practical computation of a compressible fluid, the generalized conservative forms of the metrics are consistently reduced to the well-known forms, i.e., symmetric and asymmetric conservative forms of the spatial metrics and Jacobian [23,26,7], and the symmetric form is newly introduced into the time metrics in Section 3.2. The sufficient condition of a linear high-order finite-difference scheme is arranged in detail, for which the discretized conservative metrics and Jacobian satisfy the GCL identities on moving and deforming grids in Section 3.3. Subsequently, the discretized geometric interpretation of the metrics and Jacobian is investigated using a generalized linear high-order finite-difference scheme in Section 4. In Section 5, some computational experiments are conducted to examine the computational effect of the spatial symmetry property on the metrics and Jacobian for three-dimensional stationary and deforming grids. Finally, Section 6 concludes this paper.

2. Role of the GCL identities in conservation laws

In this section, the coordinate systems and governing equations are introduced in terms of a coordinate transformation holding a strong conservation form for the governing equations. First, some notation is introduced in the manner of differential calculus and linear algebra as follows:

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[1] This identity is called the "surface (area) conservation law" in previous studies [6,26,31]. Here we refer to it as the "surface closure law" which states that the sum of the surface area vectors equals 0.