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Least squares finite element method with high continuity NURBS basis for incompressible Navier–Stokes equations

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ABSTRACT

Modern least squares finite element method (LSFEM) has advantage over mixed finite element method for non-self-adjoint problem like Navier–Stokes equations, but has problem to be norm equivalent and mass conservative when using C^0 type basis. In this paper, LSFEM with non-uniform B-splines (NURBS) is proposed for Navier–Stokes equations. High order continuity NURBS is used to construct the finite-dimensional spaces for both velocity and pressure. Variational form is derived from the governing equations with primitive variables and the DOFs due to additional variables are not necessary. There is a novel *k*-refinement which has spectral convergence of least squares functional. The method also has same advantages as isogeometric analysis like automatic mesh generation and exact geometry representation. Several benchmark problems are solved using the proposed method. The results agree well with the benchmark solutions available in literature. The results also show good performance in mass conservation.

1. Introduction

Least squares finite element method (LSFEM) has several advantages over classical Galerkin type finite element method in numerical solution of Stokes or Navier–Stokes equations [1]. Most notably, inf–sup (also known as LBB [2]) conditions are satisfied naturally, hence one can use equal order interpolation for velocity and pressure. Furthermore, the resulting algebra system always has a symmetric positive definite (SPD) coefficient matrix even for non-self-adjoint problems. Such linear equations can be efficiently solved by iterative method like preconditioned conjugate gradient. As a result, LSFEM has drawn considerable attention in the past few years [3].

The idea of LSFEM is to minimize unconstrained convex least squares functional defined as the sum of the governing equations residuals measured in some norm (mostly L^2). It is crucial to define a functional which can induce an equivalent energy norm to H^1 norm in the design of LSFEM [4,5]. When the functional is norm equivalent, the numerical solution can be interpreted as the orthogonal projection with respect to H^1 norm and is optimally accurate in H^1 norm. But norm equivalence generally involves C^1 continuous finite element space which is considered as the major drawback of LSFEM compared with weak Galerkin formulation.

Motivated by using C^0 element, the governing equations are recast as first order system by introducing auxiliary variables in modern LSFEM. For Stokes or Navier–Stokes equations, the most popular choice is velocity–vorticity–pressure formulation. This is first presented by Jiang [1,6] and also studied by Bochev [3–5,7], Pontaza [8,9] and Ozcelikkale [10], etc. There is only one additional variable in 2D case; but three additional variables are necessary in 3D case. Another choice is to introduce stress as independent variable and leads to velocity–stress–pressure formulation [11,12]. There are three additional

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variables for 2D case, and six for 3D case in this formulation. The third choice is to introduce velocity gradient which leads to velocity–gradient–velocity–pressure formulation suggested by Cai [13] and Bochev [14,15]. There are four additional variables for 2D case, and nine for 3D case in this formulation.

Each of the three formulations has some disadvantages in norm equivalence. Although the vorticity based formulation has wide acceptance in LSFEM, it fails to define norm equivalent functional for particular sets of boundary conditions [4,5]. The stress based formulation is always not norm equivalent, no matter of what kind of boundary conditions. For velocity gradient based formulation, a norm equivalent functional can be defined if the entries of the velocity gradient are constrained by incompressibility constraint. However this is only appropriate for Stokes equations, and cannot be extended to the Navier–Stokes equations. To achieve norm equivalent, there are some other efforts in literature such as weighted LSFEM [5,16], negative norm LSFEM [17,18], FOSLL* [19–21]. All of these methods have some successful application but not widely used.

The loss of norm equivalence is a result of exclusion of C^1 type element. The studies on C^1 type LSFEM are rare in literature considering the difficulty and complexity. J.P. Pontaza and J.N. Reddy [8] implement C^1 practicality in the following two ways; one is using tensor products of one-dimensional C^1 basis, the other is using discontinuous least squares functional where non-conformal finite element spaces are taken and the jump of velocity and pressure are minimized in least squares sense. However the former is sensitive to mesh distortion, and the latter lead to discontinuous velocity and pressure field.

In this paper we use non-uniform B-spline (NURBS) to achieve C^1 regularity across inter-element boundaries. NURBS is the standard and well studied tool in computer aided geometry design (CAGD) [22–24] where geometry is represented by a linear combination of independent NURBS basis. This is similar to what is done in FEM/LSFEM where unknown field is represented by a set of piecewise polynomials. The most significant difference between NURBS basis and piecewise polynomials is that, with NURBS basis, high order inter-element continuity can be achieved in an easy way. So the requirement of C^1 regularity can be fulfilled by introducing NURBS basis in LSFEM, and a norm equivalent functional can be defined. Actually the continuity of NURBS is controllable by stable and efficient order elevation algorithm [24].

We adopt the idea proposed by Hughes et al. [25,26] that is the domain and unknown fields are represented by the same basis, namely NURBS. It is analogous to isoparametric concept in FEM/LSFEM and referred as Isogeometric analysis (IGA). We name the least squares method with NURBS basis as Least Squares Isogeometric Analysis (LSIGA). NURBS is preferred to B-spline because shapes like circle, parabola and hyperbola can be represented exactly which are common in applications. There are some other gains beside norm equivalence. Mesh can be generated and refined automatically during analysis by knot insert algorithm [24] which is most attractive for adaptive methods and design optimization. There is a novel k-refinement method other than h-refinement method and p-refinement method which consists of increasing continuity of basis function and is proved to be more efficient [27,28].

The paper is outlined as follows. The least squares formulation with primitive variables for Navier–Stokes equations is given in Section 2. Finite-dimensional spaces with NURBS are constructed for 2D problems in Section 3. The method is verified by some benchmark problems in Section 4. We draw the conclusion in Section 5.

2. Least squares formulation for Navier–Stokes equations

2.1. The incompressible Navier–Stokes equations

Let $\Omega \subset \mathbb{R}^d$ be an open bounded region, where $d = 2$ or 3 is space dimension; and $\bar{\Omega} = \Omega \cup \partial\Omega$ be the closure of Ω, where $\partial\Omega = \Gamma$ is the boundary of Ω. Let $\mathbf{x}$ be a point in $\bar{\Omega}$ which has 2 or 3 components written as (x, y) or (x, y, z). The Navier–Stokes equations that govern viscous incompressible flow in Ω are specializations of momentum and mass conservation, written as:

$$-\nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} \quad \mathbf{x} \in \Omega, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \mathbf{x} \in \Omega, \tag{2}$$

where $\mathbf{u}$, p, and $\mathbf{f}$ are velocity, pressure, and body force respectively. ν is viscosity coefficient of fluid.

It is considered the inhomogeneous velocity boundary condition

$$\mathbf{u} = \mathbf{g} \quad \mathbf{x} \in \Gamma, \tag{3}$$

and zero mean pressure constraint

$$\int p\, d\Omega = 0. \tag{4}$$

Mean pressure constraint is necessary because the pressure is unique up to an additive constant. The other way to fix the constant is specify a reference pressure at one point.

2.2. Least squares formulation

In the following, we utilize the standard notation and definition for the Sobolev spaces $H^s(\Omega)$ and $H^s(\Gamma)$, where $s \geqslant 0$. The corresponding inner products are denoted as $(\cdot,\cdot)_{s,\Omega}$ and $(\cdot,\cdot)_{s,\Gamma}$; similarly the corresponding norms are denoted as $\|\cdot\|_{s,\Omega}$ and $\|\cdot\|_{s,\Gamma}$ respectively. The product spaces $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^n$, where n is the number of dependent variables, are constructed in the usual way.