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Review of summation-by-parts schemes for initial–boundary-value problems

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ABSTRACT

High-order finite difference methods are efficient, easy to program, scale well in multiple dimensions and can be modified locally for various reasons (such as shock treatment for example). The main drawback has been the complicated and sometimes even mysterious stability treatment at boundaries and interfaces required for a stable scheme. The research on summation-by-parts operators and weak boundary conditions during the last 20 years has removed this drawback and now reached a mature state. It is now possible to construct stable and high order accurate multi-block finite difference schemes in a systematic building-block-like manner. In this paper we will review this development, point out the main contributions and speculate about the next lines of research in this area.

1. Introduction

The research on Summation-By-Parts (SBP) schemes was originally driven by applications in flow problems, including turbulence and wave propagation. The objective was to use highly accurate schemes to allow waves and other small features to travel long distances, or persist for long times. One of the ground-breaking papers showing the benefit of high-order finite-difference methods for wave propagation problems is [1]. However, it has until recently proven difficult to show the same benefit in realistic simulations. Although it is easy to derive high-order finite difference schemes in the interior of the domain, it is non-trivial to find accurate and stable schemes close to boundaries. Furthermore, complicated geometries necessitate multi-block techniques. This poses yet another challenge for high-order finite difference schemes since solutions in different blocks must be glued together in a stable and accurate way. The stencils near boundaries and block interfaces create difficulties. We will focus on the so-called Simultaneous-Approximation-Term (SAT) technique where the boundary and interface conditions are imposed weakly.

The fundamental idea of SBP-SAT schemes is to allow proofs of convergence for linear and linearized problems. Convergence proofs form the bedrock of numerical analysis of PDEs since they provide the mathematical foundation that gives credibility to a numerical simulation. Without a proof of convergence, there is no guarantee that the numerical solution has any value at all. The confidence that a discrete solution is an approximation of the true mathematical solution is crucial, not only in practical engineering simulations, but also for the possibility to evaluate the accuracy of the model (i.e. the

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governing equation) itself and propose improved models. Without this quality assurance it is impossible to distinguish between modeling errors and numerical errors.

The SBP finite difference operators were first derived in [2,3] and approximate coefficients calculated. In [4], the analysis was revisited and exact expressions for the finite difference coefficients were obtained. However, the SBP finite difference operators alone, only admit stability proofs for very simple problems, and the use was limited. This changed with [5] when Simultaneous Approximation Terms were proposed to augment SBP schemes. These are penalty-like terms that enforce boundary conditions weakly. With both SBP operators and the SAT technique at hand, stability proofs for more complicated systems of partial differential equations (PDEs) were within reach.

Finite difference methods are by no means the only choice of high-order schemes. There are numerous other high-order methods with different strengths and weaknesses. However, finite difference schemes are often favored in cases where curvilinear multi-block grids can be generated, due to simpler coding and more efficient use of computer resources. For aerodynamic applications where most of the surface of the aircraft is smooth, this methodology is especially suitable since i) curvilinear grids can be generated, and ii) the resolution of large normal-to-surface gradients force the use of structured grids anyway. For very complicated geometries (such as close to landing gears), one can use hybrid methods (a combination of high-order finite differences and an unstructured method) as will be discussed below. Hybrid methods are also preferable in situations where waves propagate in free space after being generated by complicated geometrical features.

In this article, we will review the progress made towards stable high-order finite difference schemes for fluid dynamics as well as other applications. To this end, we will briefly explain the basic principles in a few examples. We will also discuss the SBP-SAT interpretation of other schemes and recent extensions of SBP-SAT schemes for time integration, non-linear theory and shock capturing.

The article is organized as follows. In Section 2, we present the theory for linear initial–boundary-value problem. We introduce the SBP-SAT concepts via a number of examples in Sections 3.1, 3.2 and 3.3. In Section 3.4 we discuss convergence rates and in Section 3.5 alternative ways to impose boundary conditions. In Sections 3.6 and 3.7 we explain the SBP-SAT method in a 2-D example. In Section 3.8 we discuss aspects of the time evolution of the discrete system and in Section 3.9 we review results regarding dual consistency. In Section 4 we relate the SBP theory for finite difference schemes to other numerical methods. Section 5 contains a review of the various applications where SBP-SAT schemes have been used to obtain numerical approximations. Finally, we discuss some aspects of non-linear theory in Section 6.

2. Theory for initial boundary value problems

We begin by reviewing the general theory for Initial–Boundary-Value Problems (IBVP). Most of the material in this section can be found in [6]. This sets the scene for the subsequent sections focusing on SBP-SAT schemes.

2.1. Preliminaries

Consider the initial–boundary-value problem

$$\begin{aligned} u_t &= P(x,t,\partial_x)u + F, \quad 0 \leqslant x \leqslant 1,\ t \geqslant 0, \\ u(x,0) &= f(x), \\ L_0(t,\partial_x)u(0,t) &= g_0(t), \\ L_0(t,\partial_x)u(1,t) &= g_1(t), \end{aligned} \tag{1}$$

where $u = (u^1, \ldots, u^m)^T$ and P is a differential operator with smooth matrix coefficients. L_0 and L_1 are differential operators defining the boundary conditions. $F = F(x,t)$ is a forcing function.

Definition 2.1. The IBVP (1) with $F = g_0 = g_1 = 0$ is well-posed, if for every $f \in C^\infty$ that vanishes in a neighborhood of $x = 0, 1$, it has a unique smooth solution that satisfies the estimate

$$\left\|u(\cdot,t)\right\| \leqslant K e^{\alpha_c t}\|f\| \tag{2}$$

where K, α_c are constants independent of f.

A problem is well-posed if it satisfies an estimate like (2). This require that appropriate boundary conditions are used which, along with the estimate, guarantees that a unique smooth solution exists. The extension to inhomogeneous boundary condition is possible via a transformation $\tilde{u} = u - \Psi$ where $\Psi(x,0) = f(x)$ and $\Psi(\{0,1\},t) = g_{0,1}$ such that $\tilde{u}$ satisfies (1) with homogeneous data (and a different but smooth forcing function). However, to obtain Ψ, $g_{0,1}$ is required to be differentiable in time. This requirement is not necessary if the problem is strongly well-posed as defined below.