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# Computation of Maxwell singular solution by nodal-continuous elements

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ABSTRACT

In this paper, we propose and analyze a nodal-continuous and $H^1$-conforming finite element method for the numerical computation of Maxwell's equations, with singular solution in a fractional order Sobolev space $H^r(\Omega)$, where $r$ may take any value in the most interesting interval $(0, 1)$. The key feature of the method is that mass-lumping linear finite element $L^2$ projections act on the curl and divergence partial differential operators so that the singular solution can be sought in a setting of $L^2(\Omega)$ space. We shall use the nodal-continuous linear finite elements, enriched with one element bubble in each element, to approximate the singular and non-$H^1$ solution. Discontinuous and nonhomogeneous media are allowed in the method. Some error estimates are given and a number of numerical experiments for source problems as well as eigenvalue problems are presented to illustrate the superior performance of the proposed method.



## 1. Introduction

In electromagnetism, the governing spatial partial differential operators are mainly the curl operator and the divergence (div) operator. As a mathematical foundation of electromagnetism, the well-known Maxwell's equations are the set of partial differential equations in terms of such two operators [15]. In general, these two characteristic operators behave rather differently from the gradient operator, although they are closely related with the latter. In fact, whenever the physical domain is nonsmooth, with re-entrant corners and/or edges on the boundary, the former would lead to singular solution of not being in the Sobolev space $H^1(\Omega)$, which is a Hilbert space of square integrable functions as well as the gradients. As a matter of fact, the singular solution belongs to a fractional order Sobolev space $H^r(\Omega)$ only, where the index $r$ which stands for the regularity of the solution may take any value in the real interval $(0, 1)$. The $H^r(\Omega)$ space is an intermediate between the $L^2(\Omega)$ space and the $H^1(\Omega)$ space, where the $L^2(\Omega)$ space is a Hilbert space of square integrable functions. This case with singular solution is also particularly relevant in discontinuous, anisotropic and nonhomogeneous media. In most cases, the regularity of Maxwell's solution is the one of the solution of the elliptic problem of Laplacian minus one, while the latter is well known in [33] to be less than two on nonsmooth domains (say, nonconvex polygons), so that the former is less than one. For more details, we refer the readers to, e.g., [2,19,22–24] and the references cited therein.

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For such a low regularity solution, it is well known that the classical nodal-continuous finite element method fails in the plain curl/div formulation. The failure exhibits an incorrect convergence. In other words, the finite element solution converges, but it does not converge to the true solution in $H^r(\Omega)$ space with $r < 1$, but to a member of $H^1(\Omega)$ space instead; see [4,10,36,40]. Such a strange phenomenon had been puzzling to the community of both mathematicians and engineers for a long while. Especially, an attempt is in vain to capture the unbounded singularity of the solution by using more elements and finer meshes near the re-entrant corners where the singularity takes place. It turns out that, for $r < 1$, the $H^1(\Omega)$ space is not dense in $H^r(\Omega)$ under the plain curl/div formulation. That is to say, the plain curl/div formulation accounts for the failure. Actually, the plain curl/div formulation would result in a Dirichlet integral formulation for any $H^1$ functions (cf. Remark 2.1 below). The Dirichlet integral of the nodal-continuous (and $H^1$-conforming) finite element solution would enforce a convergence to a member of $H^1(\Omega)$ space; see [3,4,7,9,12,18,35].

This fact directs the way out to the modification of the plain curl/div formulation. Indeed, if the nodal-continuous finite element problem is properly formulated, a correct approximation can be achieved. However, unexpectedly, until the last decade, several theoretically and numerically successful nodal-continuous finite element methods have been available, such as the weighted method [21], the $H^{-1}$ least-squares method [6,8,11], the weighted dual-potential least-squares method [39], the weighted mixed method [14], and the $L^2$ projection method [25,26,28,29,27]. These methods were designed for different models from electromagnetism. The central idea for all these is to modify the plain curl/div formulation in either the continuous stage or the discrete stage. The resultant modification can reduce or even remove the actions on the solution from the curl and the div partial derivatives. With the effects from the curl and the div operators being weaken, the nodal-continuous finite element solution could correctly converge to the true and singular solution. Note that, the resultant curl/div formulation with the modifications will no longer lead to a Dirichlet integral (cf. Remark 2.1 below), even if the finite element function is nodal-continuous and $H^1$-conforming.

In this paper we shall propose a generalization of the $L^2$ projection method to Maxwell's equations in two-dimensional bounded domain $\Omega$, of the form

$$\mathbf{curl}\, \mu^{-1} \operatorname{curl} \mathbf{u} - \varepsilon \nabla \operatorname{div} \varepsilon \mathbf{u} - \lambda \varepsilon \mathbf{u} = \varepsilon \mathbf{f}. \tag{1.1}$$

The coefficients $\mu$ and $\varepsilon$ may be discontinuous, anisotropic and nonhomogeneous, and the domain $\Omega$ may be nonsmooth with re-entrant corners and/or edges on the boundary. For such a system of Maxwell's equations, the solution may not have the $H^1$-regularity. The key technique is still to apply finite element $L^2$ projections to the curl and div operators, so that the solution can be sought in a setting of $L^2(\Omega)$ space. In essence, this type of $L^2$ projections mimics the distributional partial derivatives in the finite element spaces. Thus, the partial derivatives of the curl and div operators are transferred to the test functions. As is well known (see [16,17]), the nodal-continuous finite element space is dense in any $L^2(\Omega)$ space and even in any $L^1(\Omega)$ space (a Sobolev space of Lebesgue integrable functions). Consequently, we can expect that the underlying nodal-continuous finite element method could produce an approximation of the singular solution in $H^r(\Omega)$ with $r < 1$. Note that the $H^r(\Omega)$ space is a trivial subspace of the $L^2(\Omega)$ space.

In the present paper, we develop this new nodal-continuous finite element method for (1.1) with suitable boundary conditions. In addition to the introduction of the mass-lumping linear finite element $L^2$ projections to the curl and div operators, we shall employ the nodal-continuous linear elements, enriched with one element bubble in each element (see, e.g., [30,31,37] and the references cited therein). We should remark that this approach is essentially a three-node nodal-continuous linear finite element method, since the element bubbles can be eliminated statically in advance (cf. Appendix A). As will be seen from a number of numerical experiments, this new method is capable of approximating the singular solution in $H^r(\Omega)$ space, where $r$ can be any value in the most interesting interval $(0, 1)$. The new method is also suitable for discontinuous and nonhomogeneous media. In such cases, the solution would be prevalently more singular. In general, only some piecewise $H^r$-regularity can be available. We will provide error estimates for the case $\lambda < 0$, in which convergence and error bounds are established in the $L^2$ norm. This case was not dealt with before and the argument for this case can embody the most essential ingredients of the theory of the $L^2$ projection method.

Finally, we emphasize that there are many essential differences between the present nodal-continuous finite element method and our previous works [25–29]. In [25], we adopted local $L^2$ projections for both curl and div operators and Maxwell's solution is required to lie in $H^r(\Omega)$ for $r > 1/2$. The work [28] studies discontinuous media, adopting local $L^2$ projection for curl operator while mass-lumping $L^2$ projection for div operator. Again, the regularity $r > 1/2$ of Maxwell's solution is necessary in the error analysis. In addition, [27] focuses on the homogeneous media and [26] studies the first-order curl–div magnetostatic problem with continuous media. Notice that most of the above-mentioned works did not consider the associated eigenproblems. In contrast, [29] is devoted to study the eigenproblems, using local $L^2$ projections for both div and curl operators. However, the method in [29] still requires the singular eigenfunctions lying in $H^r(\Omega)$ for $r > 1/2$ and the discontinuous media are not studied.

The remainder of this paper is organized as follows. In Section 2, we recall the continuous problem in curlcurl–graddiv form, together with the plain curl/div variational formulation. Several representative models from computational electromagnetism are also briefly reviewed. The nodal-continuous finite element method is defined in Section 3, where two mass-lumping $L^2$ projections and the nodal-continuous finite element spaces are introduced. Error estimates for the case $\lambda < 0$ are provided in Section 4. Numerical results are presented in Section 5, with applications to the source and eigenvalue problems of Maxwell's equations, in homogeneous as well as discontinuous nonhomogeneous media. Finally, some concluding remarks are given in Section 6.