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An asymptotic-preserving scheme for linear kinetic equation with fractional diffusion limit

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A B S T R A C T

We present a new asymptotic-preserving scheme for the linear Boltzmann equation which, under appropriate scaling, leads to a fractional diffusion limit. Our scheme rests on novel micro–macro decomposition to the distribution function, which splits the original kinetic equation following a *reshuffled* Hilbert expansion. As opposed to classical diffusion limit, a major difficulty comes from the *fat tail* in the equilibrium which makes the truncation in velocity space depending on the small parameter. Our idea is, while solving the macro–micro part in a truncated velocity domain (truncation only depends on numerical accuracy), to incorporate an integrated tail over the velocity space that is beyond the truncation, and its major component can be precomputed once with any accuracy. Such an addition is essential to drive the solution to the correct asymptotic limit. Numerical experiments validate its efficiency in both kinetic and fractional diffusive regimes.

1. Introduction

The linear Boltzmann equation describes the time evolution of particle distribution that undergoes a free transport and collision with the background. The distribution function $f(t,x,v)$ depending on time $t>0$, position $x\in\mathbf{R}^N$ and velocity $v\in\mathbf{R}^N$ solves:

$$\partial_t f + v\cdot\nabla_x f = \mathcal{L}(f), \qquad (t,x,v)\in(0,\infty)\times\mathbf{R}^N\times\mathbf{R}^N, \tag{1.1}$$

$$f(0,x,v) = f_0(x,v), \tag{1.2}$$

where the collision $\mathcal{L}$ takes the form

$$\mathcal{L}(f) = \int_{\mathbf{R}^N} \left[\sigma(x,v,v')f(t,x,v') - \sigma(x,v',v)f(t,x,v)\right] \mathrm{d}v'.$$

Here $\sigma(x,v,v')\ge 0$ is the transition probability, which, under some classical conditions [9,16] gives rise to a unique equilibrium function $\mathcal{F}(v)\ge 0$ satisfying

$$\mathcal{L}(\mathcal{F}) = 0, \qquad \mathcal{F}(v) = \mathcal{F}(-v), \qquad \int_{\mathbf{R}^N} \mathcal{F}(v)\,\mathrm{d}v = 1 \quad \text{for all } x\in\mathbf{R}^N. \tag{1.3}$$

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Let ϵ be the ratio of mean free path over the macroscopic length scale, we rescale the space variable $\tilde{x} = \frac{x}{\epsilon}$ and time variable $\tilde{t} = \frac{t}{\theta(\epsilon)}$, where $\theta(\epsilon)$ that satisfies $\lim_{\epsilon\to 0}\theta(\epsilon) = 0$ will be specified later. Then equation (1.1) in the dimensionless form, upon suppression of tildes, reads

$$\theta(\epsilon)\partial_t f + \epsilon v \cdot \nabla_x f = \mathcal{L}(f). \tag{1.4}$$

Typically when the mean free path is small and time scale is large, i.e., $\theta(\epsilon) = \epsilon^2$, the distribution of the particles is assumed to be at equilibrium given by a Maxwellian distribution function, and density evolution solves a diffusion equation

$$\partial_t \rho - \nabla_x (D \nabla_x \rho) = 0,$$

where D is a diffusion matrix

$$D = \int_{R^n} v \otimes \mathcal{L}^{-1}(v\mathcal{F})\, \mathrm{d}v.$$

For an extensive review of these studies, the reader is referred to [2,3,9,14]. However, if the equilibrium is not a Maxwellian but rather a heavy tail function, i.e.,

$$\mathcal{F}(v) \sim \frac{\kappa_0}{|v|^{N+\alpha}}, \qquad 1 < \alpha < 2, \qquad \text{as } |v| \to 0,$$

the classical diffusion theory fails because the diffusion matrix D is infinite. Instead, we should consider a different time scale $\theta(\epsilon) = \epsilon^\alpha$, in which case the limit behavior is governed by a fractional (anomalous) diffusion equation. Such an equilibrium arises in numerous areas of applications such as granular plasmas with dissipative collision [5,4], astrophysical plasmas [18] and economy [10]. A rigorous derivation is undertaken in [17] via the Fourier–Laplace transform and extended in [16] for a more general space dependent or anisotropic scattering based on a weak formulation and particular choice of test function. It is revisited in a more recent work [1] following a Hilbert expansion approach which is capable in proving strong convergence results.

As opposed to sound analytical investigations, numerically solving the equation with scaling (1.4) is still at its infancy. Our goal in this paper is to provide a numerical scheme for the linear Boltzmann equation (1.4) efficient in both kinetic ($\epsilon \sim O(1)$) and anomalous diffusive ($\epsilon \ll 1$) regimes. More precisely, we want a scheme designed for the kinetic equation (1.4) preserves the asymptotic limit at the discrete level such that it automatically becomes a macroscopic solver for the fractional diffusion equation (2.4), (2.17) as $\epsilon \to 0$. This is the so called asymptotic-preserving (AP) schemes put forth by Jin [12]. A natural thought is to use implicit treatments on stiff terms, but this is impractical due to the non locality of the collision operator and the presence of the stiff convection. As such a difficulty has already appeared in the classical diffusive scaling, one might refer to some kinds of decompositions such as even–odd decomposition [13] or macro–micro decomposition [15,6,7] to separate the non-stiff part from the stiff part. However, since our limit is obtained through a *reshuffled* Hilbert expansion (the gain term in the collision switches with the convection term), previous methods do not apply. Here we propose a variant decomposition for the distribution f which splits equation (1.4) in a way that bares analogy with the reshuffled Hilbert expansion. Besides stiffness, another major difficulty lies in the fat tail which renders any truncation in velocity space inaccurate since the tail plays a significant role for small ϵ. Our idea consists of adding a tail to the decomposition, and evaluate its average value over the whole velocity space, whose major component can be precomputed once with any accuracy. We would like to point out a related work by Crouseilles–Hivert–Lemou [8] in which they consider the same problem but using a very different approach.

The rest of the paper is organized as follows. In the next section we give a brief review on the basic scaling of (1.4) and how it leads to the fractional limit. Section 3 is devoted to the major part of our scheme when the scattering is space inhomogeneous. The key ideas include: a system decomposition, a velocity truncation, and a tail compensation. In Section 4, we extend the scheme to a space inhomogeneous scattering with a simplification based on the idea of changing of space variable. Numerical examples are given in Section 5 to illustrate the efficiency, accuracy and AP properties of the new scheme. Finally the paper is concluded in Section 6.

2. Scaling and fractional diffusion limit

In this section, we specify the assumptions on the transition probability and review in brief the fractional diffusion limit of (1.4) under the appropriate choice of time scale $\theta(\epsilon)$. The rigorous theory has been carried out in [1,16,17] using various approaches such as the Laplace–Fourier transform, moment method and Hilbert expansion. We only pick up the latter approach as it facilitates our numerical scheme.

Rewrite the collision into a more convenient form

$$\mathcal{L}(f) = \int_{\mathbf{R}^N} \phi(x, v, v') \left[\mathcal{F}(v) f(t, x, v') - \mathcal{F}(v') f(t, x, v)\right] \mathrm{d}v' := K(f)\mathcal{F} - \nu(x, v) f,$$

where $\phi(x, v, v')$ is the scattering cross-section satisfying $\phi(x, v, v') = \phi(x, v', v)$. Then the gain term $K(f)\mathcal{F}$ and the loss term $\nu(x, v) f$ are defined as