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An adaptive finite volume solver for steady Euler equations with non-oscillatory *k*-exact reconstruction

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ABSTRACT

In this paper, we present an adaptive finite volume method for steady Euler equations with a non-oscillatory *k*-exact reconstruction on unstructured mesh. The numerical framework includes a Newton method as an outer iteration to linearize the Euler equations, and a geometrical multigrid method as an inner iteration to solve the derived linear system. A non-oscillatory *k*-exact reconstruction of the conservative solution in each element is proposed for the high order and non-oscillatory behavior of the numerical solutions. The importance on handling the curved boundary in an appropriate way is also studied with the numerical experiments. The *h*-adaptive method is introduced to enhance the efficiency of the algorithm. The numerical tests show successfully that the quality solutions can be obtained smoothly with the proposed algorithm, i.e., the expected convergence order of the numerical solution with the mesh refinement can be reached, while the non-oscillation shock structure can be obtained. Furthermore, the mesh adaptive method with the appropriate error indicators can effectively enhance the implementation efficiency of numerical method, while the steady state convergence and numerical accuracy are kept in the meantime.

1. Introduction

High-order numerical methods for computational fluid dynamics (CFD) have been becoming one of the most attractive research areas in CFD. A lot of pioneer works have been done towards this direction, see [31] for finite difference methods, [25,36,19] for finite volume methods (FVM), [3,23] for discontinuous Galerkin methods (DGM), [10,38] for spectral volume methods (SVM), [14,1] for residual distribution schemes, etc.

In [34], the high-order methods are defined as the numerical methods with third-order or above numerical accuracy based on a questionnaire survey among lots of professional researchers in CFD. This is not surprising since the most popular and reliable methods used in the commercial software are still first- and second-order ones. Although high-order methods have been proven their potential on delivering more efficient calculations and more accurate numerical solutions, more numerical issues need to be taken care for a successful high-order numerical scheme, compared with the low-order methods. For example, a polygon approximation for the computational domain with curved boundaries is enough for the low-order numerical methods from the numerical error point of view. However, such rude approximation will pollute the high-order

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behavior of the high-order methods. In fact, it might destroy the reliability of the numerical solution sometimes even for the low-order numerical methods, see [3]. The situation is even worse when we are considering the non-smooth solutions of governing equations, i.e., the solution containing shock or/and contact discontinuity. How to effectively remove the numerical oscillation around the discontinuities is one of the key issues in such situation. Of course the first-order methods can handle this issue perfectly. However, their intrinsic large dissipation makes them far away from the high-resolution methods, which are desirable in CFD community. Locally enriching the order of the approximation polynomial may effectively enhance the solution resolution. However, in the meantime, locally high-order approximation will bring the increment of the total variation of the solution, i.e., the non-physical oscillation of the solution will appear around the discontinuities in the domain. Theoretically, there is no hope to design linear total variation diminishing (TVD) numerical schemes with second-order or higher numerical accuracy, see Godunov theorem [35]. To overcome this, the slope limiter methods have been introduced to make the numerical method monotone. For the second-order case, a variety of slope limiters are designed to avoid the non-physical oscillation, which are mature and widely used in the commercial software, see [35,16,4] and the references therein for more details. In [15], the weighted essentially non-oscillatory (WENO) reconstruction is introduced in the linear reconstruction of the solution in the numerical framework of using finite volume methods to solve the steady Euler equations. The performance of WENO reconstruction is excellent, i.e., the convergence behavior agrees with the theoretical result very well, while the sharp, non-oscillatory shock structures are obtained.

Although there have been several reliable limiters for second-order numerical schemes, it is still a nontrivial challenge for designing an effective limiter for high-order numerical methods, especially for high dimensional problems and unstructured mesh cases. In [30,6,7], the total variation bounded (TVB) limiters have been developed towards this direction, and excellent numerical results can be observed there. Recently, lots of work have been done for using WENO methodology as a limiter for high-order numerical schemes such as [27,29,22] for DGM, [19] for FVM, and [14] for residual distribution methods (RDM). In [20,21], the authors proposed an idea of using hierarchical reconstruction method to construct the polynomial in each element, i.e., calculating the coefficients of the polynomial stage by stage. The hierarchical reconstruction is independent of shapes of the mesh elements, and is compact since only the second-order reconstruction is involved in each reconstruction stage. Consequently, the limiters for second-order reconstruction mentioned above can be used in each stage. In [37,14], the WENO reconstruction procedure is introduced in such hierarchical reconstruction method, and excellent results are presented there. Although the performance of the WENO methods is impressive in the simulations, the high demanding of the methods on the computational resource restrains their practical applications. To resolve this issue, in [28], "trouble cell" strategy is introduced to reduce the demanding on the computational resource. In each element, a smoothness indicator is defined, then the trouble elements are marked in terms of the smoothness indicators, and the WENO reconstruction will only be implemented in the "trouble cells". With this strategy, the computational resource can be saved effectively.

An alternative way to enhance the implementation efficiency of the high-order numerical methods is to use the mesh adaption. In most cases, the shock or contact discontinuity region is quite small compared with the whole domain. Hence, the dense mesh grids are only needed around discontinuities, and coarse mesh could work well elsewhere. The mesh adaptive methods can serve this purpose well. With certain kind of error indicator in every element, the mesh elements will be refined or coarsened locally, and the resulting reasonable distribution of the mesh grids will give quality numerical solution and efficient implementation. Two key issues for mesh adaptive methods are 1) how to design an efficient and reliable error indicator, and 2) how to realize the mesh adaptive method efficiently. In [18,13], a numerical framework of using h-adaptive methods to solve two dimensional steady Euler equations has been developed. To handle the mesh refinement and coarsening efficiently, a fork-tree data structure is employed. To design the error indicator, some physics based quantities are tested in the simulations. For example, the gradient of the pressure is used to generate the error indicator, which can detect the shock effectively, see [18]. The entropy based error indicator is used in [13], which works very well in the isentropic flow simulations.

In this paper, we will follow the numerical framework proposed in [13] to develop a high-order adaptive finite volume method for inviscid steady Euler equations. The numerical framework consists of two main components, i.e., a Newton iteration method as an outer iteration to linearize the Euler equations, and a geometrical multigrid method as an inner iteration to solve the derived linear system. To achieve the third-order numerical accuracy while keeping the numerical solution away from the numerical oscillation, a non-oscillatory 2-exact solution reconstruction method is developed, which is a new development of the algorithm compared with the work in [13]. The ingredients of reconstruction include the 2-exact solution reconstruction in each element, and WENO reconstruction procedure is introduced to remove the non-physical oscillation. The numerical tests show that our method successfully delivers the numerical solutions with high-order accuracy when the solutions in the domain are smooth, and non-oscillatory numerical solutions when there are shocks in the domain. The importance on handling the curved boundary in an appropriate way is studied through the numerical experiments. The results show that although our algorithm is not sensitive to the polygon approximation for the domain with complex geometry in a lot of tests, the correction for such rude approximation is necessary for some tests such as ringleb flow problem. To save the computational resource, we will follow [18,13] to employ the h-adaptive method. The numerical tests show the ability of our mesh adaptive method on saving computational resource, compared with the fixed mesh cases.

The rest of the paper is arranged as follows. In Section 2, the two dimensional inviscid steady Euler equations and the classical Godunov scheme for solving conservation laws will be briefly introduced. In Section 3, the non-oscillatory 2-exact solution reconstruction method will be presented in detail. The h-adaptive method and other numerical issues in the implementation will be stated in Section 4. In Section 5, a variety of numerical experiments will be implemented to