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High order eigenvalues for the Helmholtz equation in complicated non-tensor domains through Richardson extrapolation of second order finite differences

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ABSTRACT

We apply second order finite differences to calculate the lowest eigenvalues of the Helmholtz equation, for complicated non-tensor domains in the plane, using different grids which sample exactly the border of the domain. We show that the results obtained applying Richardson and Padé–Richardson extrapolations to a set of finite difference eigenvalues corresponding to different grids allow us to obtain extremely precise values. When possible we have assessed the precision of our extrapolations comparing them with the highly precise results obtained using the method of particular solutions. Our empirical findings suggest an asymptotic nature of the FD series. In all the cases studied, we are able to report numerical results which are more precise than those available in the literature.

1. Introduction

Among the different methods for estimating the eigenvalues and eigenfunctions of the Laplacian on a finite region of the plane, the finite difference (FD) method is the simplest, although the accuracy of the results obtained with this method is limited. In particular, for domains with reentrant corners with an angle of π/α, it is well known that the error of the FD eigenvalues is dominated by a behavior $h^{2\alpha}$ for $h \to 0$ (h is the grid spacing).

The so-called L-shaped membrane [$\alpha = 2/3$] is a famous example which was studied long time ago by Fox, Henrici and Moler [14]. Because of the quite slow convergence of FD in this case ($\Delta E \approx h^{4/3}$), those authors applied an alternative method, the method of particular solutions (MPS), and, exploiting all the symmetries of the problem, they were able to obtain the first 8 digits of the lowest eigenvalue of the L-shape correctly, $E_1 \approx 9.6397238$. Interestingly, the paper also mentions a precise (unpublished) value obtained by Moler and Forsythe, $E_1 \approx 9.639724$, extrapolating the FD values obtained with very fine grids. Unfortunately, the extrapolation is neither named nor explained.

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A valuable discussion of the Richardson extrapolation of FD results for the eigenvalues of the Laplacian on two-dimensional regions of the plane is contained in [22], where it is pointed out that the correct exponents of the asymptotic behavior of E_1 for $h \to 0$ must be used in the extrapolation.

The purpose of the present paper is to show that it is possible to obtain quite precise approximations to the eigenvalues of the Laplacian on a certain class of two-dimensional domains (specifically domains whose borders are sampled by the grid) by Richardson extrapolation of the FD results, provided that the asymptotic behavior of the FD eigenvalues for $h \to 0$ is taken into account correctly.

The paper is organized as follows: in section 2 we provide a general discussion of Richardson extrapolation, and its relation to the "method of deferred corrections"; in section 3, we describe the practical implementation of the Richardson extrapolation used in this paper; in section 4 we present the numerical results obtained for different domains, comparing them with the best results available in the literature; finally, in section 5 we summarize our findings and discuss possible directions of future work.

2. Richardson extrapolation

Richardson extrapolation is interpolation of samples of a sequence S_n by a continuous function of a continuous variable z followed by extrapolation to $z = 0$ to approximate the limit of the sequence. The slowly convergent series $\sum_{n=1}^{\infty} n^{-2}$, for example, can be summed by taking the sequence of partial sums, $S_\nu = \sum_{n=1}^{\nu} n^{-2}$, to be samples of a function in $z \equiv 1/\nu$. In our application, the sequence is that of approximations to an eigenvalue by finite difference calculations whose asymptotic error is a series in some power of the grid spacing h; here $z = h^2$ [usually] or $z = h^{4/3}$ [for one singular application].

The history including many independent discoveries is reviewed by Brezinski [7], Marchuk and Shaidurov [25], Sidi [34], Walz [39] and Joyce [21]. Christian Huyghens applied Richardson extrapolation to estimate π to 35 decimals from the perimeters of a sequence of polygons with more and more sides inscribed in the unit circle. Richardson's (1927) paper [30] contained a plethora of examples that was the first comprehensive display of the power of extrapolation; he claimed no novelty but credited others including an obscure Russian language paper by Bogolouboff and N. Krylov,[1] Richardson extrapolation of eigenvalues is discussed in Pryce's book on numerical solution of Sturm–Liouville problems [28].

Richardson extrapolation has four steps. First, compute samples $\{f(h_n)\}$ of the function being extrapolated. Second, choose a set of basis functions $\{\phi_j(x)\}$ – usually polynomials – for an approximation

$$f_N(h) \equiv \sum_{j=1}^{N} a_j \phi_j(h) \tag{1}$$

The coefficients a_j can always be computed by solving a matrix problem at a cost of $O(N^3)$ operations, and this is necessary when the ϕ_j are a mixture of polynomials and polynomials multiplied by powers of $\log(x)$, for example. However, it is faster to use Neville–Aitken interpolation to compute a *two-dimensional* array ("Richardson Table") of approximations of different N formed from different subsets of the full sample set $\{f(h_n)\}$. This is cheaper than matrix-solving [$O(N^2)$ floating point operations] though this is only a small virtue because of the speed of modern laptops. More important, extrapolation is credible only if its answers are independent of numerical choices such as N and subsets of the full set of samples. More precisely, a numerical answer is believable if and only if several different values of the numerical parameters yield the same answer to within the user chosen tolerance. The Richardson Table allows a quick search for such stable approximations. We shall return to this in analyzing each numerical example.

Various conventions are employed. A popular one is to arrange the table as a lower triangular matrix with N samples of $f(z)$, the function being approximated, as the first column:

$$R_{j,1} = f(z_j) \tag{2}$$

The simple recursion is

$$R_{j,k} = \frac{(z - z_{j-k-1})R_{j,k-1} - (z - z_j)R_{j-1,k-1}}{z_j - z_{j-k+1}}, \qquad k = j, (j+1), \ldots, N, \;\; j = 1, 2, \ldots, N \tag{3}$$

Each entry in column k is a polynomial of degree $(k-1)$ which interpolates a subset of k samples. The basic step combines two polynomials that interpolate $(k-1)$ points each to generate a polynomial that interpolates at the k points $\{z_{j-k+1}, \ldots, z_j\}$. Both generators interpolate at the $(k-2)$ points $\{z_{j-k+1}, \ldots, z_j\}$, but only $R_{j,k-1}$ interpolates at z_j while $R_{j-1,k-1}$ does not, but interpolates at z_{j-k+1}. It is easy to verify that

[1] N. Bogolouboff and N. Krylov, On the Rayleigh's principle in the theory of the differential equations of the mathematical physics and upon the Euler's method in the calculus of variations, Acad. des Sci. de l'Ukraine, Classe, Phys. Math., tome 3, fasc. 3 (1926).