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Formulation of discontinuous Galerkin methods for relativistic astrophysics

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ABSTRACT

The DG algorithm is a powerful method for solving pdes, especially for evolution equations in conservation form. Since the algorithm involves integration over volume elements, it is not immediately obvious that it will generalize easily to arbitrary time-dependent curved spacetimes. We show how to formulate the algorithm in such spacetimes for applications in relativistic astrophysics. We also show how to formulate the algorithm for equations in non-conservative form, such as Einstein's field equations themselves. We find two computationally distinct formulations in both cases, one of which has seldom been used before for flat space in curvilinear coordinates but which may be more efficient. We also give a new derivation of the ALE algorithm (Arbitrary Lagrangian–Eulerian) using 4-vector methods that is much simpler than the usual derivation and explains why the method preserves the conservation form of the equations. The various formulations are explored with some simple numerical experiments that also investigate the effect of the metric identities on the results. The results of this paper may also be of interest to practitioners of DG working with curvilinear elements in flat space.

1. Introduction

In relativistic astrophysics, simulations involving hydrodynamics or magnetohydrodynamics or similar physics are most often carried out using finite-volume methods. Two major challenges of such simulations are *accuracy* and *computational efficiency*. Many important problems cannot be solved to the required accuracy using currently available hardware resources. Accuracy can be improved only by increasing numerical resolution. If parts of the solution are smooth so that one might want to take advantage of high-order methods to improve the accuracy, current methods eventually run into problems. High-order finite-volume methods couple together more cells and require more communication between cells. Ultimately, when the number of cells and processors gets large enough, the communication time begins to limit the computation and the code no longer scales with the number of processors. Moreover, astrophysical applications often involve multiphysics (fluids, magnetic fields, neutrinos, electromagnetic radiation, relativistic gravity). With current formulations, each new type of physics often requires its own computational treatment, making coupling of the physics difficult. As one looks ahead to the arrival of exascale computing, it seems that we should look to the development of algorithms that can take advantage of these very large machines properly for astrophysics.

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In the last decade, discontinuous Galerkin (DG) methods have emerged as the leading contender to achieve all the goals of a general purpose simulation code, particularly for equations in conservation form: high order accuracy in smooth regions, robustness for shocks and other discontinuities, scalability to very large machines, accurate handling of irregular boundaries, adaptivity, and so on. Many applications of DG in terrestrial fluid dynamics have appeared. However, applications in relativity and astrophysics have so far been mainly exploratory [1–10] and confined to simple problems.

The goal of this paper is to formulate the DG method for arbitrary 3-dimensional problems involving general relativistic gravitation. At first sight, this sounds tricky: the basic DG algorithm involves integrating the pdes over space and using Gauss's Theorem to turn integrals of divergences into surface integrals. In general relativity, spacetime is curved and coordinates are arbitrary and not necessarily simply related to physical measurements by an observer. So should integrations be performed over coordinate volume or proper volume? What are the corresponding normal vectors that enter into the interface flux prescriptions? How should Einstein's equations, which are not typically in conservation form, be handled? Is the weak form or the strong form of the equations better? How do the so-called metric identities affect the formulation? We give answers to these questions. In particular, we find that the final formulation is very close to that already developed for Euclidean space in curvilinear coordinates. Moreover, the covariant approach adopted in this paper gives new insights into the standard curvilinear coordinate treatment. Not only are derivations much simpler, but alternative formulations that may be more efficient computationally are found.

Here we summarize the key results in this paper.

- Despite the curvature of spacetime, the DG algorithm can easily be formulated in general relativity. In fact, the formulation is analogous to that for curvilinear coordinates in flat spacetime.
- In the general case, there are two distinct strong formulations for conservation laws. For the tensor-product basis functions used in this paper, the corresponding weak formulations are both equivalent to one of the strong formulations.
- Only one of the formulations has been widely used for flat space in curvilinear coordinates. In numerical experiments, the other appears to be somewhat more efficient and should be further investigated.
- Similarly, there are two inequivalent formulations for hyperbolic equations in non-conservation form. These formulations are important for solving Einstein's equations.
- Time-dependent mappings (Arbitrary Lagrangian–Eulerian (ALE) methods and the dual-frame approach [11] that has proved useful for black hole simulations) are easily implemented in the relativistic treatment.
- We give streamlined derivations of the so-called metric identities, the geometric conservation law, and the ALE method for moving grids. The derivation of the ALE method is novel and uses general covariance to get the result in a few lines. In addition, the reason that the ALE method preserves the conservation form of the equations is explained.
- Satisfying the metric identities discretely is often claimed to be a necessary condition for "free-stream preservation," or the requirement that a uniform flow remain uniform for all time. We show that in fact this statement is true for only one of the computational formulations of the DG algorithm and not the other.
- We clarify how normal vectors should be normalized. The normal vector that the boundary flux vector is projected along does not need to be the unit normal — the normalization factor cancels out of the algorithm.

A covariant treatment of DG in general relativity has previously been given by Radice and Rezzolla [4]. This paper covers many aspects that were not covered by them.

2. DG for equations in conservation form

2.1. Form of the equations

In a general time-dependent curved spacetime, a conservation law can be written in terms of a 4-divergence:

$$\nabla_\mu F^\mu = 0, \tag{2.1}$$

where ∇_μ denotes the covariant derivative. Here and throughout, repeated indices are summed over. Greek indices $\mu, \nu, \dots$ range from 0 to 3, while Latin indices $a, b, \dots$ will be purely spatial, ranging from 1 to 3. We choose units with the speed of light $c = 1$, so that $x^0 = t$. We will often denote F^0 by u, a quantity like density that is conserved. The spatial flux vector F^a is generally a function of u. In practice, rather than a single conservation law like (2.1), one deals with a system of conservation laws. In this case, u is a vector of conserved quantities and F^a is a vector of flux vectors. For example, u and F^a are vectors of length 5 for hydrodynamics. In this paper, we will typically not need to deal with the various separate equations in a system of conservation laws. Accordingly, we will write u and F^a whether we are dealing with one equation or a system. All the derivations go through independently on each equation in a system.

It will be convenient to generalize (2.1) to allow a source term s on the right-hand side. Such a source term arises, for example, when one considers conservation of energy and momentum in a general curved (or curvilinear) metric. The divergence of the energy–momentum tensor gives an extra term that cannot be included as the pure divergence of a flux vector. However, the extra term depends only on u and not on its derivatives. This is the key requirement that we place on the source term in the subsequent treatment.