Journal of Computational Physics 242 (2013) 1–23

Contents lists available at SciVerse ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Riemann solver for a kinematic wave traffic model with discontinuous flux ☆

Jeffrey K. Wiens [a,*], John M. Stockie [a], JF Williams [a]

[a] Department of Mathematics, Simon Fraser University, 8888 University Drive, Burnaby, BC, Canada V5A 1S6

ARTICLE INFO

Article history:
Received 26 July 2012
Received in revised form 26 January 2013
Accepted 12 February 2013
Available online 26 February 2013

Keywords:
Hyperbolic conservation law
Discontinuous flux
Traffic flow
Lighthill–Whitham–Richards model
Finite volume scheme
Zero waves

ABSTRACT

We investigate a model for traffic flow based on the Lighthill–Whitham–Richards model that consists of a hyperbolic conservation law with a discontinuous, piecewise-linear flux. A mollifier is used to smooth out the discontinuity in the flux function over a small distance $\epsilon \ll 1$ and then the analytical solution to the corresponding Riemann problem is derived in the limit as $\epsilon \to 0$. For certain initial data, the Riemann problem can give rise to *zero waves* that propagate with infinite speed but have zero strength. We propose a Godunov-type numerical scheme that avoids the otherwise severely restrictive CFL constraint that would arise from waves with infinite speed by exchanging information between local Riemann problems and thereby incorporating the effects of zero waves directly into the Riemann solver. Numerical simulations are provided to illustrate the behavior of zero waves and their impact on the solution. The effectiveness of our approach is demonstrated through a careful convergence study and comparisons to computations using a third-order WENO scheme.

1. Introduction

In the 1950's, Lighthill and Whitham [34] and Richards [40] independently proposed the first macroscopic traffic flow model, now commonly known as the LWR model. Although this model has proven successful in capturing some aspects of traffic behavior, its limitations are well-documented and many more sophisticated models have been proposed to capture the complex dynamics and patterns observed in actual vehicular traffic [25]. Despite this progress, the LWR model remains an important and widely-used model because of its combination of simplicity and explanatory power.

The LWR model consists of a single scalar nonlinear conservation law in one dimension

$$\rho_t + f(\rho)_x = 0, \tag{1}$$

where $\rho(x,t)$ is the traffic density (cars/m),

$$f(\rho) = \rho v(\rho)$$

is the traffic flow rate or flux (cars/s), and $v(\rho)$ is the local velocity (m/s). The most commonly used flux function is

☆ This work was supported by two Discovery Grants from the Natural Sciences and Engineering Research Council of Canada (NSERC).

* Corresponding author.
E-mail addresses: jwiens@sfu.ca (J.K. Wiens), jstockie@sfu.ca (J.M. Stockie), jfwillia@sfu.ca (JF Williams).
URL: http://www.math.sfu.ca/~stockie (J.M. Stockie).

0021-9991/$ - see front matter
http://dx.doi.org/10.1016/j.jcp.2013.02.024

$$f(\rho) = u_{max}\rho\left(1 - \frac{\rho}{\rho_{max}}\right), \quad (2)$$

which was obtained by Greenshields [23] in the 1930's by fitting experimental measurements of vehicle velocity and traffic density. Here, u_{max} is the maximum free-flow speed, while ρ_{max} is the maximum density corresponding to bumper-to-bumper traffic where speed drops to zero. The LWR model belongs to a more general class of *kinematic wave traffic models* that couple the conservation law Eq. (1) with a variety of different flux functions.

Extensive studies of the empirical correlation between flow rate and density have been performed in the traffic flow literature. This correlation is commonly referred to as the *fundamental diagram* and is represented graphically by a plot of flux f versus density ρ such as that shown in Fig. 1. A striking feature of many experimental results is the presence of an apparent discontinuity that separates the free flow (low density) and congested (high density) states, something that has been discussed by many authors, including [9,17,18,26]. In particular, Koshi et al. [27] characterize flux data such as that shown in Fig. 1 as having a *reverse lambda* shape in which the discontinuity appears at some peak value of the flux.

This behavior is also referred to as the *two-capacity* or *dual-mode phenomenon* [3,4] and has led to the development of a diverse range of mathematical models. Zhang and Kim [45] incorporated the capacity drop into a microscopic car-following model that generates fundamental diagrams with the characteristic reverse-lambda shape. Wong and Wong [42] performed simulations using a multi-class LWR model from which they also observed a discontinuous flux-density relationship. Colombo [11] developed a macroscopic model that couples an LWR equation for density in the free flow state, along with a 2×2 system of conservation laws for density and momentum in the congested state; the phase transition between these two states is a free boundary that is governed by the Rankine–Hugoniot conditions. This work has been extended upon by Goatin [20], Chalons and Goatin [10], and Blandin et al. [5]. Lu et al. [35] incorporated a discontinuous (piecewise quadratic) flux directly into an LWR model, and then solved the corresponding Riemann problem analytically by constructing the convex hull for a regularized continuous flux function that consists of two quadratic pieces joined over a narrow region by a linear connecting piece.

There remains some disagreement in the literature regarding the existence of discontinuities in the traffic flux, with some researchers (e.g., Hall [24]) arguing that the apparent gaps are due simply to missing data and can be accounted for by providing additional information about traffic behavior at specific locations. Indeed, Persaud and Hall [39] and Wu [43] contend that the discontinuous fundamental diagram should be viewed instead as the 2D projection of a higher dimensional smooth surface.

We will nonetheless make the assumption in this paper that the fundamental diagram is discontinuous. Our aim here is not to argue the validity of this assumption in the context of traffic flow, since that point has already been discussed extensively by [35,42,45], among others. Instead our objective is to study the effect that such a flux discontinuity has on the analytical solution of a 1D hyperbolic conservation law, as well as to develop an accurate and efficient numerical algorithm to simulate such problems.

A related class of conservation laws, in which the flux $f(\rho, x)$ is a discontinuous function of the spatial variable x, has been thoroughly studied in recent years (see [7,8,46–48] and references therein). Considerably less attention has been paid to the situation where the flux function has a discontinuity in ρ. Gimse [19] solved the Riemann problem for a piecewise linear flux function with a single jump discontinuity in ρ by generalizing the method of convex hull construction Ch. 16 [32]. In par-

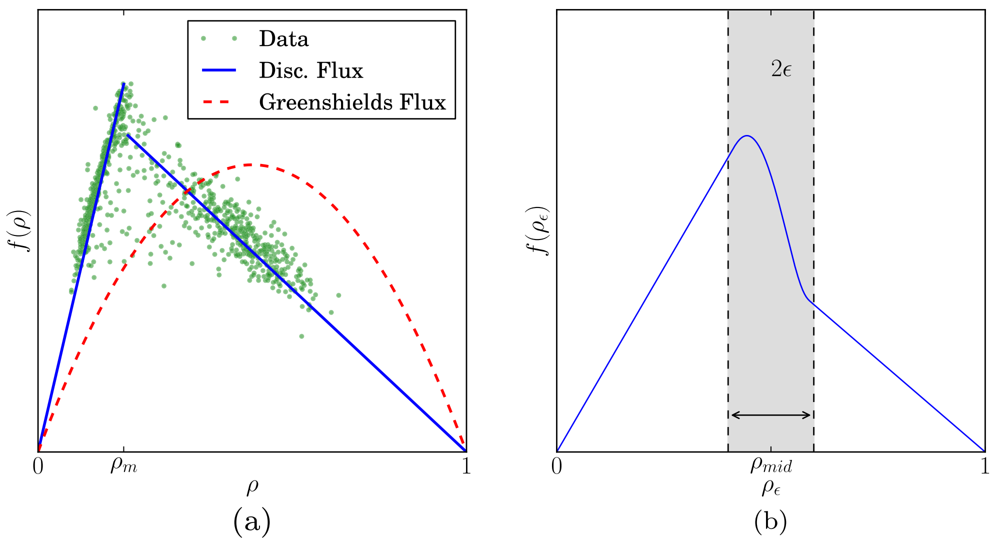

Fig. 1. In (a), the discontinuous "reverse lambda" flux function Eq. (4) is overlaid with empirical data extracted from [24] Fig. 7 (reproduced with permission of Elsevier B.V.), along with the quadratic Greenshields flux (2). The mollified flux from Eq. (7) is depicted in (b).