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Hermite WENO schemes for Hamilton–Jacobi equations on unstructured meshes ☆

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ABSTRACT

In this paper, we extend a class of the Hermite weighted essentially non-oscillatory (HWENO) schemes for solving the Hamilton–Jacobi equations by Qiu and Shu (2005) [24] to two dimensional unstructured meshes. The idea of the reconstruction in the HWENO schemes comes from the original WENO schemes, however both the function and its first two derivative values are evolved via time advancing and used in the reconstructions, while only the function values are evolved and used in the original WENO schemes which are nodal based approximations. The third and fourth order HWENO schemes using the combinations of second order approximations with nonlinear weights and TVD Runge–Kutta time discretization method are used here. Comparing with the original WENO schemes for Hamilton–Jacobi equations, one major advantage of HWENO schemes presented here is its compactness in the reconstructions. Extensive numerical tests are performed to illustrate the capability and high order accuracy of the methodologies.

1. Introduction

In this paper, we adopt the first order monotone Hamiltonian developed by Abgrall [2] in conjunction with the Hermite WENO (HWENO) reconstruction methodology on unstructured meshes. And then, it is used to solve the Hamilton–Jacobi (HJ) equations as:

$$\begin{cases} \phi_t + H(\nabla\phi) = 0, & (x, y, t) \in \Omega \times [0, \infty), \\ \phi(x, y, 0) = \phi_0(x, y), & (x, y) \in \Omega, \end{cases} \tag{1.1}$$

where $\nabla\phi = (\phi_x, \phi_y)^T$.

The Hamilton–Jacobi equations are often used in geometric optics, computer vision, material science, image processing and variational calculus [6,17,25]. Yet, the solutions to (1.1) are continuous but their derivatives can be discontinuous. And such solutions may not be unique unless using the physical implications and then getting the viscosity solutions [3]. It is well known that the HJ equations are closely related to conservation laws, hence successful numerical methods for conservation laws can be adapted for solving the HJ equations. Among successful numerical methods for conservation laws, we mention the essentially non-oscillatory (ENO) and weighted ENO (WENO) methods. The ENO methodology can be tracked back to the work proposed by Harten and Osher [8]. The key idea of ENO schemes is applying the most smooth stencil among all candidate stencils to approximate the solutions at cell boundaries to a high order of accuracy and avoid

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oscillations near discontinuities. In 1994, Liu et al. [18] proposed a WENO scheme that was constructed from the $(r+1)$-th order ENO schemes to obtain $(r+2)$-th order accuracy. Then in 1996, Jiang and Shu proposed the framework to construct finite difference WENO schemes from the $(r+1)$-th order (in L^1 norm sense) ENO schemes to get $(2r+1)$-th order accuracy, gave a new way of measuring the smoothness, and emulated the ideas of minimizing the total variation of the approximation in [12]. The first ENO and WENO papers on unstructured meshes were proposed by Abgrall in 1994 [1] and Friedrich in 1998 [7], respectively. The application of the WENO ideas and the whole reconstruction algorithm were also described in detail. And then, Shu and Hu gave two robust WENO schemes [9] which could maintain high order of accuracy by using a combination of lower algebraic polynomials in the finite volume formulation on unstructured meshes. The key idea of WENO schemes is applying all the stencils to approximate the variables at cell boundaries to an even higher order of accuracy in smooth regions and also can avoid oscillations near discontinuities. WENO type schemes can improve ENO type schemes in many aspects [27].

The framework of the ENO and WENO schemes is to evolve only one degree of freedom per cell, namely the cell average for the finite volume version or the point value at the center of the cell for the finite difference version. In [22,23,30,31], a class of WENO schemes based on Hermite polynomials, termed HWENO schemes, were presented for solving one and two dimensional nonlinear hyperbolic conservation law systems. The main difference between the Hermite WENO scheme designed in [22,23,31] and the traditional WENO schemes is that the former has a more compact stencil than the latter for the same order of accuracy. This compactness is achieved by evolving both the function and its first derivative values in time and they are both used in the reconstruction in HWENO schemes. Numerical examples in [22,23,30,31] demonstrate that HWENO schemes work well for solving hyperbolic conservation laws.

Based on the facts that the HJ equations are closely related to the conservation laws, many numerical methods for the conservation laws can be amended and used for solving the HJ equations. Along this line, Osher and Sethian [19] proposed a second order ENO scheme and Osher and Shu [20] presented high order ENO schemes to solve the Hamilton–Jacobi equations. Then, a high order WENO scheme was proposed by Jiang and Peng [11]. HWENO schemes for solving the Hamilton–Jacobi equations on structured meshes were presented by Qiu and Shu [21,24]. In 1996, Lafon and Osher [14] constructed the ENO schemes for solving the Hamilton–Jacobi equations on unstructured meshes. Zhang and Shu [29], Li and Chan [16] further developed high order WENO schemes for solving two dimensional Hamilton–Jacobi equations by using the nodal based weighted essentially non-oscillatory algebraic polynomial reconstructions on triangular meshes. And some finite element methods for arbitrary triangular meshes were developed in [4,5,10,15].

In this continuation paper, we would extend HWENO method to solve the Hamilton–Jacobi equations on unstructured meshes. We evolve both the viscosity solution ϕ at the barycenter of the triangular cell and the cell averages of its derivatives ϕ_x, ϕ_y over the cell. Both the point values of the solution and the cell averages of its derivatives are used to reconstruct the point values of the derivatives at the points of different inner sectors on the triangular cell and its boundaries. Comparing with the original WENO schemes of Jiang and Peng [11], Zhang and Shu [29] and Li and Chan [16], one major advantage of HWENO schemes is its compactness in the reconstructions, since both the solution and its derivatives are evolved in time.

The organization of this paper is as follows: In Section 2, we review and construct the third and fourth order HWENO schemes in details for solving Hamilton–Jacobi equations on unstructured meshes and present extensive numerical results in Section 3 to verify the accuracy and stability of these approaches. Concluding remarks are given in Section 4.

2. The construction of Hermite WENO schemes for the Hamilton–Jacobi equations

In this section, we firstly give the framework of solving the Hamilton–Jacobi equations briefly and then develop the procedures of the third and fourth order HWENO schemes on unstructured meshes for the Hamilton–Jacobi equations in details.

2.1. The framework

We take the governing equation (1.1), in which Ω has a triangulation consisting of cells Δ_i, $i=1,\ldots,N$. We denote (x_i, y_i), $|\partial\Delta_{i_{\ell\ell}}|$, $\ell\ell=1,2,3$ and $|\Delta_i|$ to be the barycenter, the different edge length and the area of the triangular cell Δ_i, respectively. Let $u(x,y,t)=\phi_x(x,y,t)$ and $v(x,y,t)=\phi_y(x,y,t)$, and taking the x, y derivatives of (1.1), we can obtain the conservation laws:

$$\begin{cases} u_t + H(u,v)_x = 0, \\ u(x,y,0) = \frac{\partial\phi_0(x,y)}{\partial x}, \end{cases} \tag{2.1}$$

and

$$\begin{cases} v_t + H(u,v)_y = 0, \\ v(x,y,0) = \frac{\partial\phi_0(x,y)}{\partial y}. \end{cases} \tag{2.2}$$

We define $\phi_i=\phi(x_i,y_i,t)$ to be the numerical approximation to the viscosity solution of (1.1) at the barycenter of the target (triangular) cell Δ_i, the cell average of u as $\bar{u}_i(t)=\frac{1}{|\Delta_i|}\int_{\Delta_i} u(x,y,t)\,dxdy$ and the cell average of v as $\bar{v}_i(t)=$