Journal of Computational Physics 251 (2013) 1–16

Contents lists available at SciVerse ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

An arbitrary Lagrangian Eulerian method for three-phase flows with triple junction points

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ARTICLE INFO

Article history:
Received 16 January 2013
Received in revised form 14 May 2013
Accepted 15 May 2013
Available online 4 June 2013

Keywords:
Three-phase flows
Triple point
ALE
Interface
Finite element
Moving mesh
Adaptive mesh
Cubic spline
Uzawa method

ABSTRACT

We present a moving mesh method suitable for solving two-dimensional and axisymmetric three-liquid flows with triple junction points. This method employs a body-fitted unstructured mesh where the interfaces between liquids are lines of the mesh system, and the triple junction points (if exist) are mesh nodes. To enhance the accuracy and the efficiency of the method, the mesh is constantly adapted to the evolution of the interfaces by refining and coarsening the mesh locally; dynamic boundary conditions on interfaces, in particular the triple points, are therefore incorporated naturally and accurately in a finite-element formulation. In order to allow pressure discontinuity across interfaces, double-values of pressure are necessary for interface nodes and triple-values of pressure on triple junction points. The resulting non-linear system of mass and momentum conservation is then solved by an *Uzawa* method, with the zero resultant condition on triple points reinforced at each time step. The method is used to investigate the rising of a liquid drop with an attached bubble in a lighter liquid.

1. Introduction

Three phase flow problem has generated wide interest among theoretical researchers; for instance, different regimes of the floating lens problem have been analyzed in [1]. Three phase flow problem is also of importance in industrial application. Recently, a study of vertical migration of a gas-oil drop in a capillary filled with water was presented at an APS (American Physical Society) meeting in 2008 by Prof. Kent Udell; his research on this flow configuration is motivated by the extraction of dense oil in porous media. This work is devoted to a novel numerical method for three-phase flows with emphasis on the numerical treatment of triple junction points.

There is a strong contrast between the large number of publications on numerical work on two-phase flows and very little numerical results on three-phase flows. Most of numerical methods on two-phase flows in the past fifteen years are based on a volume tracking type method, such as VOF [2–4], level-set [5] and phase-field method [6]. They all use a scalar (marker) function to locate the interface. Interfaces are represented therefore *implicitly*. The interface tension is often modeled by a CSF (Continuum Surface Force) type method [7] for the VOF and level-set methods, while the phase-field method has its own built-in surface tension model. The moving contact line problem can be investigated successfully in this way [8–10]. Moving contact line is a *special* triple junction point where a gas/liquid–liquid interface meets a solid surface. As the solid surface is in general fixed, one marker function suffices to represent the gas/liquid–liquid interface.

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http://dx.doi.org/10.1016/j.jcp.2013.05.029

The moving contact line problem is difficult (for reviews, see [11,12]). The difficulty does not arise in the triple point problem considered here. The triple point is allowed to move anywhere as long as the pulling forces of the three interfaces balance each other exactly at this point (the zero resultant condition). It is quite surprising that this seemly easier triple point problem has received much less attention. The true reason may be the difficulty involved in a corresponding numerical method. The above volume tracking method is adequate for two phase flow problems where only one marker function is needed. There is no theoretical objection regarding the use of two or more marker functions to represent three phase flows, and a similar methodology as that of two phase flows works as long as the different interfaces do not meet; when they do meet, an accurate CSF type method based *implicit* representation of triple junction points is in general not an easy matter.

The numerical method developed in this work follows another philosophy of interface tracking. We go back to one of the earliest ideas, tracking the interface *explicitly* [13]. A recent work in this spirit is the grid-alignment finite element technique developed in [14]. This idea has fallen out of favor due the requirement of highly sophisticated algorithms to maintain a mesh of good quality, as well as the difficulty of dealing with the topological change of interface. However, there are many flow situations where interface deformation is moderate and explicit tracking is possible. Explicit representation of interface allows accurate approximations of boundary conditions on the interfaces. It is precisely this advantage that this work strives to explore in the study of flows with triple junction points. In this work, we investigate the triple point problem in the framework of Navier–Stokes equations. It is worthwhile to note that curvature dependent motion of triple point junctions has been studied, taking the advecting velocity as being proportional to the local curvature [15].

Numerical solutions of moving boundary problems can also be classified according to the mesh used: moving grid methods and fixed grid methods. Each method has its own advantages and disadvantages. Fixed grid methods are capable of simulating very complex interface motion [16–18]. The accuracy in the treatment of boundary conditions is an issue of this type of methods, and new algorithms have been developed to improve it [19,20]. Moving grid methods employ a so-called boundary-fitted grid system [21,22]. This type of method has the tremendous advantage that the boundary conditions can be treated neatly and resolved very accurately, because the moving boundary coincides with one line of the numerical grid. The numerical method developed in this work falls into this second category. The author has previously worked on a boundary-fitted method based on finite volume method (FVM) using structured mesh [23]. This work is based on a finite element method (FEM) using an adaptive unstructured triangulation method. It is part of our effort to incorporate complicated physics involving moving boundaries in a general framework. For instance, we intend to include new abilities, such as handling a viscoelastic fluid [24] and the Marangoni effect induced from a surfactant [25]. The FEM approach is more flexible in generating high quality meshes in more complicated situations. This flexibility is vital for the accurate treatment of the physical condition at the triple points. One disadvantage of the FEM method based on unstructured meshes is the computational cost; the computing time per vertex is in general much higher than that of a method based on structured meshes, although a carefully generated unstructured mesh can reduce the total number of mesh vertices needed, hence the efficiency weakness.

The rest of this paper is organized as follows. Section 2 details the governing equation and the boundary conditions on the interfaces, in particular, the boundary condition at the triple points. Section 3 presents the numerical method. Section 3.1 describes an adaptive mesh generator and our interface tracking method. It provides the foundation for this work. The finite element method is presented in Section 3.2. The $P_2 - P_1$ Taylor–Hood element is used in general. The conventional approach requires that the pressure is continuous all over the domain (P_1–C^0). This is however inadequate for multiphase flow computation, because the pressure across the interface may be discontinuous, for instance, the Laplace solution of a static drop resting in another reference liquid. In order to take into account pressure discontinuity, the pressure must take multiple values at interfaces and triple points. We introduce therefore a new element space, that we call discontinuous P_1d element space, for pressure. In this space, the pressure has a single-value on interior vertices, double-values on interfaces vertices, and triple-values on triple junction points. Finally, numerical results are included in Section 4.

2. Governing equations and boundary conditions

We consider flows of three different liquid phases L_i with density ρ_i and viscosity μ_i, where $i = 1, 2, 3$. The surface tension coefficient on an interface between two different liquids L_i and L_j is noted as $\sigma_{(ij)}$. The three-fluid flows are in general highly non-linear processes. They are governed by the incompressible Navier–Stokes equations. In the two-dimensional case, the continuity equation reads:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

and the momentum equation

$$\rho \frac{du}{dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\mu \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right), \tag{2}$$

$$\rho \frac{dv}{dt} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) + \frac{\partial}{\partial y}\left(2\mu \frac{\partial v}{\partial y}\right) + \rho g, \tag{3}$$