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A low Mach number solver: Enhancing applicability

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A B S T R A C T

In astrophysics and meteorology there exist numerous situations where flows exhibit small velocities compared to the sound speed. To overcome the stringent timestep restrictions posed by the predominantly used explicit methods for integration in time the Euler (or Navier–Stokes) equations are usually replaced by modified versions. In astrophysics this is nearly exclusively the anelastic approximation. Kwatra et al. (2009) [19] have proposed a method with favorable time-step properties integrating the *original* equations (and thus allowing, for example, also the treatment of shocks). We describe the extension of the method to the Navier–Stokes and two-component equations. However, when applying the extended method to problems in convection and double diffusive convection (semiconvection) we ran into numerical difficulties. We describe our procedure for stabilizing the method. We also investigate the behavior of Kwatra et al.'s method for very low Mach numbers (down to $Ma = 0.001$) and point out its very favorable properties in this realm for situations where the explicit counterpart of this method returns absolutely unusable results. Furthermore, we show that the method strongly scales over three orders of magnitude of processor cores and is limited only by the specific network structure of the high performance computer we use.

1. Introduction

In astrophysics, meteorology and many other fields of physical and engineering sciences problems are studied which are characterized by vastly different timescales. In fluid flow the smallest time scales are often set by diffusive processes, whereas the solution changes only on the much larger time scales set by the flow velocity or the sound speed. It is then general practice to make use of explicit time integration for the hyperbolic terms and to treat the diffusive terms implicitly. The latter task is facilitated by the fact that the resulting system of discretized equations is often linear and positive definite so that effective solution methods are available.

Frequently, however, the different scales originate from the macroscopic flow speed (small) and the sound speed (large). In addition, the sound waves may be known to be of little physical significance. Such a situation is ubiquitous in meteorology and geophysics (convection in the earth mantle). In stellar physics, these premises typically hold true for convection in the

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interior of stars, for example in the solar convection zone except for its outermost parts. Since an implicit time-advancement of the hyperbolic part is usually impractical due to, among others, a lack of efficient solvers for these terms, the usual solution strategy here is to eliminate the sound waves analytically and to discretize the modified equations only. A widely (in stellar physics: nearly exclusively) used way to accomplish that is to resort to the anelastic approximation [27] and its many variants or extensions. In meteorology, the use of so-called primitive equations is also widespread (cf., [1], e.g.).

In [19] Kwatra et al. propose a semi-implicit method to solve Euler's equations numerically. From a physical point of view the advantage of this approach is that the method avoids any simplifications on the side of the basic equations themselves. As a consequence of that it is possible to treat physical phenomena for which the anelastic methods are neither designed nor appropriate, e.g. shocks. Indeed, in [19] a number of shock tube problems are satisfactorily treated as test cases. Furthermore, it is outlined that this method has favorable properties in the low Mach number regime. On the numerical side, the implicit part consists of the solution of an equation of a generalized Poisson equation only (Eq. (32) in the present paper). For such equations, efficient solvers are quite readily available. In [19], the method was developed in the framework of ENO type numerics.

These properties made the method an interesting candidate for inclusion in our ANTARES code [26] which is designed to cope with flows encountered in stellar physics ranging from low to high Mach numbers, including shocks. The original development in terms of ENO methodology complied well with the fact that ANTARES makes use of this methodology also in the other methods included for spatial discretization.

While after a few adaptions the Kwatra method performed well in the usual shock tube tests we performed, difficulties arose when applying its straight forward extension to the problem of semiconvection in stellar physics (double-diffusive convection; diffusive convection in oceanography) in the same way we had, earlier on, successfully tackled similar problems with standard ENO methods. In semiconvection possibly slow (in terms of sound speed) convective motions take place. Already they enforce the calculations to span a large time interval in order to arrive at a relaxed, statistically steady state. In addition, between zones of ordinary convection thin, more or less horizontal sheets are situated where the transport of helium in the star (salt in the ocean) is provided by diffusive processes only, which may easily make for even longer time scales [35]. Experience thus showed that the successful mastering of the shock tube problems as usually applied for validating numerical discretizations does not guarantee stability for long periods of time. The extended version of Kwatra's method, which basically consisted of adding the viscous terms to the explicit part of the method did not yield stable results. Indeed, severe instabilities appeared in the course of time which rendered the calculations useless. This was quite surprising since the explicit ENO-based method employing the same discretization led to satisfyingly stable results. It was therefore warranted to stabilize the original method so as to achieve stability even for long-term integrations.

The plan of the paper is the following. We first give a short description of the basics of Kwatra et al.'s method. Subsequently, we derive a pressure evolution equation which, unlike the original equation, takes also dissipative terms into account (radiative transfer in the diffusion approximation as adequate for stellar interiors). We also consider the case of two-component flows (semi- or thermohaline convection). We then describe the difficulties which resulted when applying this method to the semiconvection problem. Next we turn to the enhancement of stability which solved the numerical problems and discuss a few simulations.

Subsequently, we show that this method scales strongly over three orders of magnitude of processor cores and is limited only by the specific network structure of the high performance method we use. This makes it suitable for high performance computing despite the necessity of solving an elliptic equation each timestep.

Furthermore, we validate the method of Kwatra et al., by testing it in the high Mach number regime as well as its performance in low Mach number flows (down to $Ma = 0.001$). In the latter regime, Kwatra's method shows very satisfying behavior which contrasts grossly with the completely useless results of the standard explicit method.

We conclude that, given the beneficial properties of this method, it poses a real alternative to the anelastic or Boussinesq approximations.

2. Numerical method

We model fluid flow by the set of compressible Navier–Stokes equations (NSE). A detailed derivation is found in [10,20]. For better legibility, all derivations are presented in one spatial dimension. Hence, $\partial/\partial x$ is, in this case, the one-dimensional divergence operator ∇. The generalization to higher dimensions is easily done.

The compressible Navier–Stokes equations for a chemically homogeneous flow read

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \rho u \\ e \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \rho u \\ \rho u \cdot u + P - \tau \\ eu + P \cdot u - u \cdot \tau - F_{\text{rad}} \end{pmatrix} = \begin{pmatrix} 0 \\ \rho g \\ \rho g u \end{pmatrix} \tag{1}$$

The state variables in the Navier–Stokes equations depend in principle on the spatial variable x and time t. The explicit dependencies are stated in Table 1. In general, the radiative source term $Q_{\text{rad}} = \nabla \cdot F_{\text{rad}}$ is determined as the stationary limit of the radiative transfer equation (see [23] for discussion and further references). However, we will only consider settings taking place in stellar regions which are optically thick, so this quantity can accurately be obtained by means of the diffusion approximation for radiative transfer, $Q_{\text{rad}} = \nabla \cdot F_{\text{rad}} = -\nabla \cdot (K \nabla T)$.