Journal of Computational Physics 236 (2013) 443–460

Contents lists available at SciVerse ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

The finite difference approximation for a class of fractional sub-diffusion equations on a space unbounded domain ☆

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ARTICLE INFO

Article history:
Received 25 April 2012
Received in revised form 15 September 2012
Accepted 20 November 2012
Available online 5 December 2012

Keywords:
Fractional sub-diffusion equation
Space unbounded domain
Finite difference scheme
Stability
Convergence
Artificial boundary conditions

ABSTRACT

In this paper, for a class of fractional sub-diffusion equations on a space unbounded domain, firstly, exact artificial boundary conditions, which involve the time-fractional derivatives, are derived using the Laplace transform technique. Then the original problem on the space unbounded domain is reduced to the initial-boundary value problem on a space bounded domain. Secondly, an efficient finite difference approximation for the reduced initial-boundary problem on the space bounded domain is constructed. Different from the method of order reduction used in [37] for the fractional sub-diffusion equations on a space half-infinite domain, the presented difference scheme, which is more simple than that in the previous work, is developed using the direct discretization method, i.e. the approximate method of considering the governing equations at mesh points directly. The stability and convergence of the scheme with numerical accuracy $O(\tau^{2-\gamma}+h^2)$ are proved by means of discrete energy method and Sobolev imbedding inequality, where γ is the order of time-fractional derivative in the governing equation, τ and h are the temporal stepsize and spatial stepsize, respectively. Thirdly, a compact difference scheme for the case of $\gamma\leqslant 2/3$ is derived with the truncation errors of fourth-order accuracy for interior points and third-order accuracy for boundary points, respectively. Then the global convergence order $O(\tau^{2-\gamma}+h^4)$ of the compact difference scheme is proved. Finally, numerical experiments are used to verify the numerical accuracy and the efficiency of the obtained schemes.

1. Introduction

Anomalous diffusion behaviors have been observed in numerous complex systems, such as in polymers, biopolymers, organisms, liquid crystals, fractals and percolation clusters, proteins and ecosystems [1–5]. The fractional differential equation is one of the effective tools describing this phenomenon. The first-order time derivative term in classical diffusion equations is replaced by the fractional derivative of order γ $(0<\gamma<2)$ and the resulting equations are called fractional anomalous diffusion equations. More precisely, when $0<\gamma<1$, it is called fractional sub-diffusion equations. And when $1<\gamma<2$, it is classified as fractional wave equations.

Recently, the fractional differential equation has been widely investigated both from the analytical and numerical viewpoints. As we known, analytical solutions of most fractional differential equations are unavailable [1–3]. Even for some

☆ This work is supported by National Natural Science Foundation of China (No. 11271068).
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http://dx.doi.org/10.1016/j.jcp.2012.11.011

specific and simple equations, their exact solutions are associated with Fox H functions. The frequently appeared form contains the Wright function, Mittag–Leffler function and hyperbolic geometry functions. These functions take the form of the series and the computation of them is not so easy. The need of engineering applications enhances the research of numerical methods for fractional differential equations, since these equations have been admitted to be the valuable tools in depicting numerous phenomenons.

The finite difference approximation for fractional differential equations is an interesting and important issue in scientific computing and engineering fields. Many works were dedicated to the investigation of the initial-boundary value problems on space bounded domains. Here we just mention some among them. For one-dimensional fractional sub-diffusion equations with the Caputo time-fractional derivatives, Zhuang and Liu [6] presented an effective difference scheme based on L1 approximation for the Caputo time-fractional derivatives and established the stability and convergence of the scheme using the discrete maximum principle, where the analyzed accuracy was only $O(\tau + h^2)$. Murillo and Yuste [7] developed the explicit finite difference schemes and showed the stability conditions by means of fractional von-Neumann techniques. Sun and Wu [8] constructed the implicit finite difference scheme using L1 formula and gave the theoretical analysis using the discrete energy method. Lin and Xu [9] presented the numerical scheme with L1 discretization for time-fractional derivatives and spectral approximation for space derivatives. The numerical scheme using spectral approximation in both time and space directions was developed by Li and Xu in [10]. In [11,12], the authors considered the finite difference approximations for fractional sub-diffusion problems with Neumann boundary conditions on space bounded domains.

The problems of integer-order on space unbounded domains have been considered by many scholars. Just as what were pointed out in [13–15], the heat problems on unbounded domains come from the heat transfer in solids, fluid dynamics, option pricing theory in financial mathematics, or other areas of applied mathematics. The core of solving problems on space unbounded domains numerically is to construct the appropriate boundaries to make the computational domains finite. Then by deriving the exact or approximate conditions on these boundaries, the original problems on space unbounded domains can be reduced to the initial-boundary value problems on space bounded computational domains. And solutions of the reduced problems coincide with that of the original problems on the computational domains. It is just the idea of artificial boundary methods (ABMs). About the comprehensive description of ABMs, the readers can look up the review articles [16,17] and the monograph [18]. Until now, the ABMs have been used to deal with masses of problems of integer-order, such as linear problems [13,15,19–22], semilinear parabolic PDEs [23], semilinear wave equations [24] and nonlinear problems (Burgers equations [25,26], Schrödinger equations [27,28], Stokes and Navier–Stokes equations [29], etc.).

As early as in 1980s, Wyss et al. [30,31] began to seek the analytical solutions of the Cauchy and Signaling problems for the fractional diffusion equations. The solutions were expressed as the form of Fox-H functions. Recently, the fractional differential equations on space unbounded domain have also been investigated in some literatures. Mainardi [32] discussed the fundamental solutions of the basic Cauchy and Signaling problems using the Laplace transform, where the solutions were expressed in terms of the M-Wright functions. Metzler and Klafter [33] investigated the solutions of fractional diffusion equations in half-space with reflecting and absorbing boundaries, respectively, where the method of images was employed. Agrawal [34] presented the general solutions for two sets of problems, namely, Cauchy problem and Signaling problem, by using the Laplace transform and Green function method. Povstenko [35] considered solutions of the Signaling problem for time-fractional diffusion-wave equation in a half-plane by using the Laplace integral transform with respect to time and the Fourier transforms with respect to spatial coordinates. The solutions were written in terms of the Mittag–Leffler functions. The same problem in the case of angular symmetry was reported in [36] using integral transform technique.

The analytical solutions of fractional diffusion-wave problem on space unbounded domain have been studied to some extent, however, the numerical methods of these problems are still in an early stage of development. Very little is known. One attempt extending the idea of ABMs for problems of integer-order to fractional differential equations on a space half-infinite domain has been achieved in [37]. In [37], the method of order reduction is used to construct the finite difference scheme for solving the reduced initial-boundary problem on the space bounded domain. One of the major advantages of this method lies in that no errors in spatial direction are produced when approximating the derivative boundary conditions. However, intermediate variables are introduced for the derivation of finite difference scheme and the analysis process of the resultant scheme also refers to these intermediate variables, although the computing of the scheme does not involve them longer usually. The obtained difference scheme is often established at middle points of the spatial mesh, i.e. the scheme refers to the averaging in the spatial direction. One more simple way can be used to construct the difference scheme for the problem. This point is one of the motivations for our present paper.

The outline of this paper is as follows. By deriving the exact artificial boundary condition, the original problem is reduced to an initial-boundary value problem on a space bounded domain in Section 2. In Section 3, a finite difference scheme for solving the reduced problem is developed by considering the governing equation directly at mesh points. The difference scheme here is simpler than the corresponding one in our previous work [37], which is obtained by means of the method of order reduction. Section 4 deals with the rigorous stability and convergence analysis of the difference scheme using the discrete energy method and the Sobolev imbedding inequality. In Section 5, a compact difference scheme for the case of $\gamma \leqslant 2/3$, as well as the convergence analysis is discussed. In Section 6, the numerical accuracy, the reliability and efficiency of the schemes are confirmed by some numerical examples. Finally, a brief conclusion of the paper is included.