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Energy-conserving Runge–Kutta methods for the incompressible Navier–Stokes equations

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ABSTRACT

Energy-conserving methods have recently gained popularity for the spatial discretization of the incompressible Navier–Stokes equations. In this paper implicit Runge–Kutta methods are investigated which keep this property when integrating in time. Firstly, a number of energy-conserving Runge–Kutta methods based on Gauss, Radau and Lobatto quadrature are constructed. These methods are suitable for convection-dominated problems (such as turbulent flows), because they do not introduce artificial diffusion and are stable for any time step. Secondly, to obtain robust time-integration methods that work also for stiff problems, the energy-conserving methods are extended to a new class of additive Runge–Kutta methods, which combine energy conservation with *L*-stability. In this class, the Radau IIA/B method has the best properties. Results for a number of test cases on two-stage methods indicate that for pure convection problems the additive Radau IIA/B method is competitive with the Gauss methods. However, for stiff problems, such as convection-dominated flows with thin boundary layers, both the higher order Gauss and Radau IIA/B method suffer from order reduction. Overall, the Gauss methods are the preferred method for energy-conserving time integration of the incompressible Navier–Stokes equations.

1. Introduction

1.1. Advantages of energy-conserving methods

In this work we address the time integration of flows governed by the incompressible Navier–Stokes equations. In case of inviscid flow with periodic or no-slip boundary conditions these continuous equations possess a number of properties, also called *symmetries* or *invariants*, see e.g. [1]. Such inviscid flows are of interest because many flows of practical importance are convection-dominated. In this paper we focus on one important invariant of inviscid incompressible flows, namely the kinetic energy. Upon discretizing the continuous equations in space and/or time this invariant is often not conserved. There are, however, several (related) reasons to have a discretization conserving energy in a discrete sense. Firstly, from a physical point of view, an energy-conserving scheme is free of numerical diffusion. This is important for turbulent flow simulations with DNS or LES, because it prevents numerical diffusion from overwhelming the molecular diffusion (in case of DNS) or the effect of the sub-grid model (in case of LES), so that the energy spectrum is not affected. Energy-conserving discretizations guarantee that all diffusion is modeled (laminar and/or turbulent), and not artificial. This is why energy-conserving schemes

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are seen as a necessity for DNS and LES by different researchers, see e.g. [2–8]. Energy-conserving methods necessitate the use of central schemes for the convective terms. Upwind schemes, typically used in RANS simulations of turbulent flows, are robust because they introduce numerical diffusion, but should for this reason not be used in LES or DNS. Even high-order upwind methods can damp turbulence fluctuations and mask the effects of the sub-grid scale models used in LES [9–11]. Although central schemes do not have numerical diffusion, they introduce dispersive errors; these were found to be less detrimental than diffusive errors, at least in the simulation of turbulent channel flow [12]. Secondly, from a more mathematical point of view, discrete energy conservation provides a non-linear stability bound to the solution (see e.g. [13]). Flow simulations are then stable for any mesh and any time step, so that these parameters can be chosen purely based on accuracy requirements. This is especially important for simulating turbulent flows that involve large time and/or length scales, like weather prediction [14]. Lastly, energy-conserving methods are important when dealing with coarse grids and large time steps. Simulations of turbulence with DNS and LES are computationally very expensive and mesh sizes are kept as large as possible in practice, even under-resolved. On the other hand, the order of a discretization scheme is defined for vanishing mesh sizes and time steps, and on coarse grids it is not obvious whether the order of a method is still a good measure of accuracy and whether a formally higher order method is preferred over a formally lower order method [15,16]. Energy-conserving methods are of particular interest then, because they lead to well-posed discrete operators and as a consequence to well-behaved global errors.

One application of energy-conserving methods that we have in mind is wind-turbine wake aerodynamics, a situation in which turbulent flow structures travel over large distances while mixing with the atmospheric boundary layer [17]. This is an example in which one cannot resolve the small scale turbulent fluctuations in detail; we believe that preserving their total energy and dissipation rate is then of utmost importance.

Discrete energy conservation requires an appropriate spatial discretization and an appropriate temporal discretization method. The focus of this paper is on the latter, but in order to provide a background for the reader, we will summarize existing work on both spatial and temporal energy conservation.

1.2. Overview of spatially energy-conserving schemes

The first energy-conserving scheme for the incompressible Navier–Stokes equations is probably Harlow and Welch's staggered grid method [18]. The staggering of the variables leads to a method that conserves mass, momentum, energy and vorticity, and strongly couples pressure and velocity, making it the method of choice for simulating incompressible flows on Cartesian grids [19]. Ham et al. [20] extended the method to retain these properties on non-uniform grids. In order to simulate flows around complex geometries, Wesseling et al. [21,19] extended the method to general structured grids that can be described by a Cartesian grid in computational space. On unstructured meshes staggered methods have been investigated by Perot and co-workers [3,22]. Mahesh et al. [2] also consider energy-conserving methods for unstructured meshes and formulate a second order staggered method for tetrahedral elements, but propose a non-staggered formulation for elements of more general shape. This results in a formulation that is not fully energy-conserving, because the pressure gradient contributes to the kinetic energy.

In fact, the contribution of the pressure gradient to the kinetic energy is intrinsic to non-staggered layouts [23,12]. Felten and Lund [12] show that this energy error makes staggered schemes superior to collocated schemes in case of inviscid simulations and in case of viscous simulations with relatively coarse meshes. However, operators on collocated meshes can be 'shifted' to obtain operators for staggered meshes, as shown by Hicken et al. [24]. This elegant approach can be seen as a generalization of the work of Perot to general meshes [3], including locally refined ones. Numerical experiments show that on anisotropic Cartesian grids, where the local truncation error of the gradient is inconsistent (zeroth order), it is still possible to have a first order accurate solution, emphasizing the positive influence of energy-conserving methods on global discretization errors.

High-order energy-conserving methods have been addressed for finite difference methods by Morinishi et al. [23] and Vasilyev [25], who indicate how to obtain any (even) order of accuracy on uniform grids based on Richardson extrapolation. On non-uniform grids strict conservation and (local) order of accuracy cannot be obtained simultaneously. Verstappen and Veldman [8,7] employ a finite volume method and construct a fourth order accurate method on non-uniform grids that retains all properties of the Harlow and Welch scheme. They call their method 'symmetry-preserving', because it is based on mimicking symmetry properties of continuous operators in a discrete sense, instead of on minimizing the local truncation error. Preserving these symmetries leads to energy conservation. Again, it should be stressed that the global order of accuracy cannot be derived from the local truncation error alone; the discretized operator is at least of equal importance, and it is indeed the energy-conserving schemes that lead to a discretized operator that is well-posed, even on coarse meshes.

Another way to obtain higher order methods is to use compact (implicit) schemes, either in finite difference or finite volume context (see e.g. [26,27] for references). Implementation of boundary conditions is easier due to their smaller computational stencil, and furthermore they have better resolution of high wave numbers than explicit schemes. Knikker [26] obtains energy conservation with a fourth order compact finite difference scheme with the nonlinear terms in skew-symmetric form, Hokpunna and Manhart [27] mention that energy conservation for compact finite volume methods is still an open issue.

Lastly, this section would be incomplete without mentioning the recent review article of Perot [28] and the work on 'mimetic' methods, like the support-operator method by Shashkov [29], Hyman et al. [30], the work on summation-by-parts